Edmund C. Stoner and the discovery of the maximum mass of white dwarfs

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The existence of a mass limit for white dwarfs is usually attributed solely to the late astrophysicist Subrahmanyan Chandrasekhar, and this limit is named after him[1]. But as is often the case, the history of this discovery is more nuanced. In this note I will show that the existence of a maximum mass was first established by Edmund C. Stoner, a physicist who began experimental research under the supervision of Rutherford at the Cavendish in Cambridge, but later switched to theoretical work. Rutherford recommended Stoner to a position at the Physics department of the University of Leeds where he spent his entire career[2]. According to G. Cantor, he was "probably the leading Cavendish-trained *theoretical* physicist of the 1920's "[3], although he learned theory mostly on his own, and became known for his work on magnetism[4]. Unfortunately, Stoner suffered from diabetes and poor health which restricted his travels, and this may account for the fact that he did not receive wider recognition for his achievements.

In 1924 Stoner wrote a paper on the distribution of electrons among atomic levels[5]. In the preface of the fourth edition of his classic book, "Atomic Structure and Spectral Lines", Arnold Sommerfeld gave special mention to " einen grossen Fortschritt [a great advancement]" brought about by Stoner's analysis, which then came to the attention of Wolfgang Pauli, and played and important role in his formulation of the exclusion principle in quantum physics [6] Therefore, it is not surprising that Stoner's interest in white dwarfs was aroused by Ralph H. Fowler's suggestion[7] [8] that the exclusion principle could be applied to solve a major puzzle, the origin of the extreme high density of white dwarfs [9] [10], which could not be explained by classical physics. Eddington expressed this puzzle as follows:

`` I do not see how a star which has once got into this compressed state is ever going to go out of it... *The star will need energy in order to cool*...It would seem that the star will be in an awkward predicament when its supply of subatomic energy fails. Imagine a body continually losing heat but with insufficient energy to grow cold ! " [11].

At the time, the conventional wisdom was that the source of internal pressure which maintained all stars in equilibrium against gravitational collapse was the internal pressure of the matter composing the star which had been heated into a gas presumably, according to Eddington, by "subatomic energy". But when this supply of energy is exhausted and the star cools, Fowler proposed that a new equilibrium would ensue, even at zero temperature, due to the "degeneracy" pressure of the electrons caused by the exclusion principle. Fowler, however, did not attempt to determine the equilibrium properties of such a star which he regarded as "strictly analogous to one giant molecule in the ground state''. Apparently he was unaware that at the time, Llewellyn H. Thomas had developed a mathematical method to solve this problem in atomic physics[12]. Subsequently, Stoner developed a novel minimum energy principle to obtain the equilibrium properties of such dense stars[13], and by applying Fowler's non-relativistic equation of state for a degenerate electron gas in a constant density approximation, he found that the density increases with the square of the mass of the star[14]. In such a gas the mean momentum of an electron is proportional to the cube root of the density (see Appendix I), and Wilhem Anderson, a privatdozent at Tartu University, Estonia, who had read Stoner's paper, noticed that for the mass of a white dwarf comparable to or higher than the mass of the Sun, the density calculated from Stoner's non-relativistic mass-density relation implied that the electrons become relativistic [15]. Hence, Anderson concluded that in this regime, this relation gave "gröblich falschen Resultaten [gross false results]" for the properties of a white dwarf. He attemped to extend the equation of state of a degenerate electron gas to the relativistic domain, but he gave an incorrect formulation which, fortuitously, indicated that Stoner's minimum energy principle implied a maximum value for the white dwarf mass. Alerted by Anderson's paper, Stoner then derived the correct relativistic equation of state[16], and re-calculated, in a constant density approximation, the properties of white dwarfs for arbitrary densities[17]. Thus, he obtained, now on solid theoretical grounds, the surprising result that when the density approaches infinity, the mass of the star reaches a *maximum* value.

Two years after the appearance of the first paper[13] by Stoner on the " limiting density of white dwarfs", Chandrasekhar published a paper[18] with a similar title "arriving at the order of magnitude of the density of white stars from different considerations". This paper was communicated by Fowler to the *Philosophical Magazine* . Since the non-relativistic pressure - density relation for a degenerate electron gas is a power law with exponent 5/3 (see Appendix I), Chandrasekhar realized - from having read Eddington's book "The Internal Constitution of the Stars"[11] , which he had obtained as an essay prize - that the solution of the differential equation for gravitational equilibrium of a low mass white dwarf was the Lande-Emde polytrope with index n=3/2. This solution leads to the same mass – density relation previously found by Stoner in the uniform density approximation, but with a proportionality coefficient smaller by a factor about two . Meanwhile, Stoner, in collaboration with Frank Tyler, also calculated the minimum energy of a white dwarf assuming a density distribution corresponding to the n=3/2 polytrope [19] obtaining the same result as Chandrasekhar , and somewhat earlier Edward A. Milne also had carried out this calculation [20]. In his paper Chandrasekhar ignored "relativistic-mass corrections", because he did not yet know how to incorporate them, while Stoner had shown that for the white dwarf companion of Sirius these corrections gave a density almost an order of magnitude larger than the non-relativistic calculation. In his recollections[21], however, Chandrasekhar remarks that he had found that the degenerate electrons become relativistic [22] for white dwarfs with masses which are comparable or larger than the mass of the Sun. His calculation in the extreme relativistic limit appeared separately in a very short paper (two pages long) on " the maximum mass of ideal white dwarfs [23]. Again, Chandrasekhar was able to obtain his result with great ease, because the relevant solution of the differential equation for gravitational equilibrium for the extreme relativistic equation of state of a degenerate electron, which has an exponent 4/3 (see Appendix I), corresponds to the the $n=3$ Lane-Emde polytrope solution, which also appears in Eddington's book [11] . It turns out that for $n=3$ the mass is *independent* of the central or mean density of the star. Chandrasekhar acknowledged that his result was in surprising "agreement" with Stoner's result , but he also claimed, without giving any proof, that the critical mass was a *maximum*. Later, in an interview with Spencer Weart [24], Chandrasekhar acknowledged that

"…at first I didn't understand what this limit meant and I didn't know how it would end [25], and how it related to the 3/2 low mass polytropes. But all that I did when I was in England and wrote my second paper on it".

But a proof that the critical mass is a maximum already had been given in the uniform density approximation by Stoner, who also had shown analytically that the mass of a white dwarf is a *monotonically* increasing function of the density which is finite at infinite density, while it took Chandrasekhar several additional months before he found a rough argument to show that at the critical mass the density becomes infinite[26]. But the fact that he was aware of Stoner's analysis was left unmentioned, although it is clear that it must have given him confidence in the validity of his result.

Stoner's fully relativistic analytic solution, in the uniform density approximation (see Appendix I), for the mass-radius dependence of the dense stars[17] is shown graphically in Fig. 1. His result is compared with ten *numerical* calculations, shown by circles, which Chandrasekhar obtained *five* years later by integrating numerically the differential equations of gravitational equilibrium with Stoner's relativistic pressure-density equation of state[27].

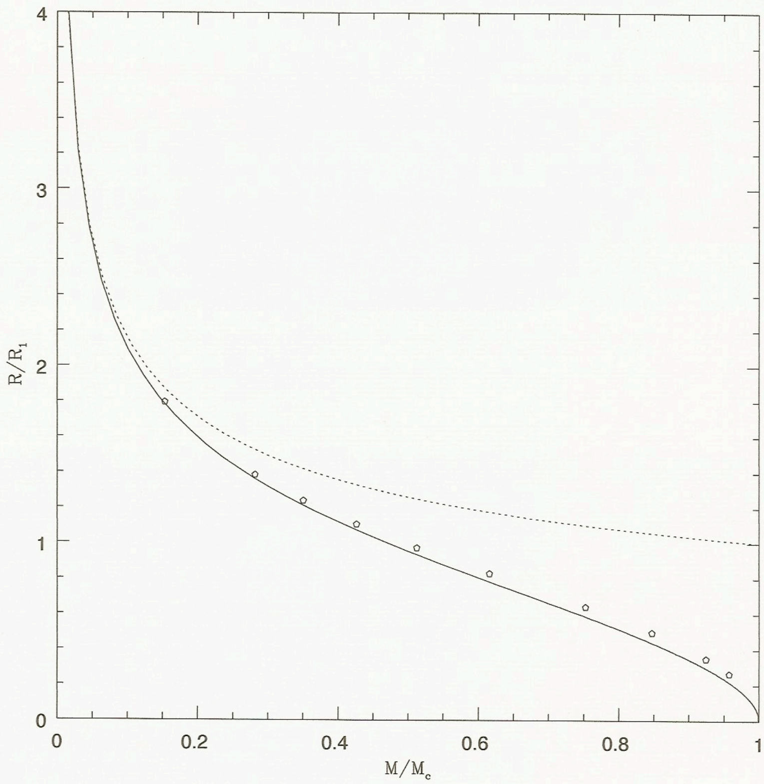


Fig. 1 The dark line is a plot of the scaled radius, R/R_1 vs. scaled mass, M/M_c of Stoner's 1930 *analytic* solution in the uniform density approximation (see Appendix I) . The circles are the solutions published in 1935 by Chandrasekhar, who *numerically* integrated the equations of gravitational equilibrium using Stoner's pressure-density relativistic equation of state (see Appendix I). The mass is given in units of the critical mass M_c, and the radius in units of a length R_1 for which $(M/M_c)(R/R_1)^3 \approx 1$ in the non-relativistic limit, $(M/M_c) << 1$. The dashed line is the non-relativistic solution $R/R_1 = (M_C/M)^{1/3}$.

This remarkable agreement is surprising, because Stoner's result was based on the uniform density approximation, while Chandrasekhar's was obtained by integrating the equations of gravitational equilibrium. The main difference is in the scales of mass and of length, e.g. Chandrasekhar's critical mass M_c is 20 % smaller that Stoner's. Before 1935, following ideas of Milne[20], Chandrasekhar had developed only a crude composite model for a white dwarf [26] in which the non-relativistic approximation was assume to be valid for increasing mass until the central pressure became equal to the pressure given by the extreme relativistic equation at the same density. For a larger mass, he applied this relativistic equation to a central region of the star, and the non-relativistic equation for an external region of the star bounded by a surface defined when these two equations gave the same pressure at equal densities.

Stoner was encouraged by Arthur S. Eddington, the foremost astrophysicist at that time, to pursue the implication of his relativistic equation of state on the maximum density and temperature of white dwarfs as a function of density, and he communicated Stoner's two papers on this subject to the Monthly Notices of the Royal Astronomical Society[28][29][30]. Eddington's 1932 correspondence with Stoner (see Appendix II and Fig. 2) deepens further the mystery why several years later, in a well known public attack on Chandrasekhar's similar work on white dwarfs [31], Eddington unexpectedly rejected the relativistic equation of state, and the profound implications of the existence of a white dwarf mass limit[32][33] for the fate of stars with masses exceeding this limit[34]. Apparently Eddington had found that relativistic degeneracy was incompatible with his fundamental theory, and later confessed to Chandrasekhar that he would have to abandon this theory if relativitivistic degeneracy was valid[35]. Eddington's criticisms [36] were entirely unfounded[37] but his enormous prestige led to the acceptance of his views by many in the astronomical community, and to an early rejection of Chandrasekhar's work. After Eddington questioned the validity of the relativistic equation of state for a degenerate electron gas, Chandrasekhar went for support to several of the great pioneers of the modern quantum theory, including Dirac who was in Cambridge, and to Bohr and Rosenfeld who he had met during a visit at Bohr's Institute in Copenhagen. They assured him of the validity of the relativistic equation of state [38], and advised him to ignore Eddington's objections [39], but Chandrasekhar continued relentlessly to pursue this matter, writing a paper with Christian MØller on relativistic degeneracy[40], and persuading Rudolf Peierls to give another proof [41] of its validity. During this controversy, however, Chandrasekhar apparently did not mention Stoner and his earlier derivation of this equation [17], which is neither referenced in his paper with MØller nor in the paper by Peierls. In an Appendix to the first paper [27] in which he applied Stoner's equation, he claimed to offer a "simpler derivation" of it, but it turned out to be essentially the same one given by Stoner. Here Chandrasekhar did give an acknowlegdment to Stoner with the remark that " this equation has been derived by Stoner (among others)", but the "others" remain unidentified, because they don't exist. He also mentioned " that Stoner had previously made some calculations concerning the (p, ρ) relations for a degenerate gas", neglecting to give reference to Stoner's paper[28] where a derivation of this pressure-density relation and his numerical tables appeared. For several more years Stoner continued to work on the equation of state for finite temperatures, publishing extensive tables of Fermi-Dirac functions [42] which later turned out to be also very useful for improved calculations of the properties of white dwarfs [43]. Chandrasekhar also did not mentioned that an independent derivation in 1931 of the critical mass of dense stars was given by Lev Landau[44], who apparently was unaware of Stoner's work. Landau, however, could not have known of Chandrasekhar's work which appeared only after Landau had submitted his work for publication. Nevertheless, in his "historical notes" [21], Chandrasekhar complained "the tendency in some current literature" to give Landau priority in this discovery, and never gave reference to Landau's work.

Later on, in his 1939 book [45] on stelllar book where he reproduced his work on white dwarfs, Chandrasekhar mentioned that the "equation for the internal energy of an electron gas" was derived by E. C. Stoner (p. 361), but again he neglected to refer to Stoner's explicitly derivation of the pressure-density relation, and his numerical tables for such a gas[28], although in 1934 he had to reproduce these tables with higher accuracy, because these tables were essential for his numerical integrations of the differential equations for gravitational equilibrium. He claimed (p. 422) that " the existence of this limiting mass was first isolated by Chandrasekhar , though its existence had been made apparent from earlier considerations by Anderson and Stoner …". One is left wondering, however, what he meant by this assertion. I have found two other occasions when he used the word "isolate'', which may give a clue to its meaning in the present context. In his book "Eddington , the most distinguished astrophysicst of this time '' (Cambridge Univ. Press, Cambridge 1983), Chandrasekhar stated that when Eddington calculated the relation between mass and pressure in a star, he did not "isolate'' its dependence on *natural constants,* "a surprising omission in view of his later preoccupations with natural constants''. Likewise, in his Nobel speech [31], Chandrasekhar remarked that an inequality, given as Eq. (14), had "isolated'' the combination of *natural constants* of the dimension of mass. But in this sense, it was Stoner and not Chandrasekhar who first "isolated'' the limiting mass, because Stoner explicitly gave the dependence of this mass on natural constants [13] (see Appendix I). In his ``Biographical Notes'' (p. 451) where he gives a reference to only two of the five papers of Stoner on the properties of white dwarfs [13] [17], Chandrasekhar's merely comments that in these papers "Stoner makes some further applications of Fowler's ideas'' , not giving the reader any idea of the important concepts and results regarding the properties of white dwarfs contained in these seminal papers. By such obfuscation, Chandrasekhar gave rise to the current neglect of Stoner's work.

In Kamesh Wali's excellent biography of Chandrasekhar [32], Stoner, is not mentioned even once, and his name also does not appear in Spencer Weart's transcript [24] of his lengthy interview with Chandrasekhar in 1977. More recently, in his book "The Empire of Stars''[33], Arthur Miller remarks that "it was indeed extraordinary that a nineteen-year-old Indian youth [Chandrasekhar] had managed to make a discovery that had eluded the great minds of European astrophysics'' (p.14) . Although Miller briefly refers to Anderson and to Stoner, he claimed that they "had never examined the ramifications'' of the relativistic equation of state (p. 133). But as we have shown here, with respect to Stoner Miller's claim is incorrect. In 1983 Chandrasekhar was awarded the Nobel prize, but in his acceptance speech, which mainly is a historical review of his work on white dwarfs, he did not include a single reference to Stoner. This general neglect of Stoner's seminal work on white dwarfs helps explain why, with a few notable exceptions [46 47 48], Stoner's contributions and his priority in the discovery of the maximum mass of white dwarfs have been forgotten now.

Summary and Conclusions

One of the primary purposes of the history of science is to understand how fundamental concepts were discovered and developed in the past . Sometimes the path is obscured by the all too human tendency of some scientists to enhance their own contributions, while neglecting to acknowledge properly the important influence of others. This is illustrated here in the case of the discovery of the limiting mass of white dwarfs. The main purpose of this study is not to assign priorities, but to show how the essential scientific developments took place.

After Fowler suggested that degeneracy pressure of the electrons was responsible for the high density of white dwarfs, both Anderson and Chandrasekhar realized, independently of each other, that for white dwarfs with masses comparable to that of the Sun the mean energy of the degenerate electrons becomes relativistic. Then Stoner and Chandrasekhar, also independently of each other, discovered that the extreme relativistic form of the equation of state for a degenerate electron gas implied the unexpected result that there is a critical mass for white dwarfs. According to Chandrasekhar's account of his discovery, which he repeated on numerous occasions [21 24 31 32 49], both Fowler and Milne were at first not interested in this result , and five years later Eddington publicly ridiculed him for engaging in "stellar buffoonery" [32 33]. This episode has become one of the best known legends in astronomy, and has been told to generations of students in this field. They have been given, however, only a partial historical account, because Stoner's important role has always been passed over in silence. Actually, the early reception of the discovery of the limiting mass also appears to have been more nuanced. When Chandrasekhar arrived in Cambridge and mentioned his discovery to Fowler, in effect Fowler responded that he had been scooped by Stoner [21]. Likewise, from references in a paper by Milne [20], it is clear that Milne also was aware of Stoner' s work, because he applied it to his own theory of stellar interiors, without, however, examining the implications of relativity. Therefore Fowler and Milne's supposed lack of interest in Chandrasekhar's account of the limiting mass may partly have been due to the fact that they did not considered it to be a novel discovery. Moreover, early on, both Milne and Eddington encouraged Chandrasekhar to do further research on the white dwarf problem, while at the same time, Eddington also encouraged Stoner to work on this problem. Surprisingly, Eddington even offered to collaborate with Stoner (see Appendix II) , who was away in Leeds, rather than with Chandrasekhar, who was at his institute in Cambridge. Evidently, Eddington recognized that Stoner could apply the fully relativistic equation of state for a degenerate electron gas at *arbitrary densities*, while, with his method, Chandrasekhar could consider only the non-relativistic (low density) and extreme relativistic (infinity density) limits. This prevented Chandrasekhar from carrying out a complete analysis of the properties of white dwarfs until five years after Stoner had done a comparable analyis in the uniform density approximation. There is no evidence that Chandrasekhar understood the relationship between his mathematical approach using the gravitational equation of equilibrium, and Stoner's minimum energy principle which I will describe below [14] .

Appendix I. Description and comparison of Stoner's and Chandrasekhar's methods

Stoner's method [13] [17] for obtaining the properties of white dwarfs was based on his concept that at equilibrium, the sum of the internal energy and the gravitational energy of the star should be a *minimum* for a fixed mass of the star. Fowler had assumed that the atoms in a white dwarf were completely ionized, and that the internal energy and pressure was entirely due to a degenerate electron gas, while the ions mainly accounted for the mass of the star [7]. Stoner understood that as the star contracts, the gravitational energy decreases, and since the density increases, the internal energy also increases. Hence, the total energy of the star either decreases or increases during the contraction of the star. By conservation of energy, when the total energy of the star *decreases*, radiation and/or other forms of energy must be emitted by the star. But without an external source of energy, the total energy of an isolated star cannot increase. Hence the contraction of the star must end if the total energy reaches a minimum, and then the star reaches an equilibrium [14] .

To calculate the density at which the total energy minimum occurs, Stoner started with an approximation by assuming that the density was uniform. In his first paper [13] he applied Fowler's non-relativistic form for the degeneracy energy, and he found that the density depends quadratically on the mass of the star. Later, in collaboration with F. Tyler [19], he also considered the modification for non-relativistic degeneracy when the density varies according to a polytrope distribution with index $n = 3/2$. Then, after Anderson [15] pointed out that for a white dwarf with a mass of the order of the mass of the Sun Stoner's analysis implied that the electrons become relativistic, Stoner obtained the general relativistic equation of state for a degenerate electron gas, and he applied it to obtain the mass-density relation of white dwarfs for *arbitrary* densities [17]. By means of his minimum energy principle, he obtained and analytic expression which gave this relation in *parametric* form, showing that the density is a function that increases monotonically, and more rapidly than the square of the star's mass. In particular, he obtained the fundamental result that the density approaches infinity for a finite mass. This is the celebrated limiting mass of white dwarfs, in which the mass scale is entirely determined by some of the fundamental constants of Nature.

Chandrasekhar's early method was based on applying the Lande-Emde polytrope solution of the differential equation for gravitational equilibrium for the equation of state of a degenerate electron gas which obey power laws in the non-relativistic and the extreme relativistic regime. He obtained results similar to Stoner's for the white dwarf mass-density relation in the non-relativistic regime [18], and for the critical white dwarf mass in the extreme relativistic regime [23]. For a power law dependence of the pressure p on the density ρ, i.e. $p \propto \rho^{\gamma}$, where the exponent γ is a constant, the solution of this equation is given by the the Lane-Emden polytrope of index n, where $\gamma = 1 + 1/n$. Chandrasekhar found these solutions in Eddington's book, *"The Internal Constitution of Stars"* [11], which also contained the relations and numerical quantities that he needed for his calculations. In the non-relativistic limit, $\gamma = 5/3$, corresponding to a polytrope with index $n = 3/2$, and this Lane-Emde solution gives the central or mean density dependence as the square of the mass of the star, the same result which Stoner had obtained two years earlier in the uniform density approximation. Substituting Fowler's non-relativistic pressure density relation, Chandrasekhar found that the magnitude of this dependence is smaller than Stoner's value by a factor approximately equal to two [18]. But somewhat earlier, motivated by Stoner's work, E. Milne already had carried out this calculation[20], and at about the same time Stoner and Tyler [19] also had applied the $n = 3/2$ polytrope density in Stoner's minimum energy method, and obtained the same result. In the *extreme* relativistic limit, $\gamma = 4/3$, the corresponding polytrope has index $n = 3$, and the mass is *independent* of the central or mean density of the star. Thus Chandrasekhar calculated the magnitude of the critical mass of white dwarfs, which depends on the fundamental constants of nature, as had been shown a year before by Stoner, and on a dimensional constant for the $n = 3$ polytrope . This gave a critical mass about 20 % smaller than Stoner's value for the uniform density approximation [18]. By his own admission, however, Chandrasekhar was puzzled by his result [21], and he was not able to show until several months later that the critical mass was a maximum , and that in this limit the density was infinite. Moreover, he did not pursue the implications of this result, and for several years he assumed that at a certain value of the density, matter would become incompressible, an idea proposed earlier by Milne to avoid infinite density at the center of his models of a star [20]. Chandrasekhar formulated this idea as follows":

"We are bound to assume therefore that a stage must come beyond which the equation of state $p = K\rho^{4/3}$ is not valid, for otherwise we are led to the physically inconceivable result that for $M = .92M_s$ [M_s =solar mass and $\mu = 2.5$], $r_1 = 0$, and $\rho = \infty$. As we do not know physically what the equation of state is that we are to take, we assume for definiteness the equation for the homogeneous material $\rho = \rho_{max}$, where ρ_{max} is the maximum density of which the material is capable...[50] "

For $M > .92M_s$ Chandrasekhar assumed that there was a homogeneous core with $\rho = \rho_{max}$ surrounded by a relativistic envelope. This required, however, an unrealistic model of the star, where the density must become discontinuous at an interface [51]. It was not until 1934 that he dropped these crude models, after visiting Ambartsumian in Moscow [32 33], who suggested that he integrate directly the equations for gravitational equilibrium by applying the full relativistic equation of state for a degenerate electron gas at arbitrary densities; in other words, that he apply Stoner's equation of state (see below) .

It is of interest to inquire what the relation is between Stoner's minimum energy method and Chandrasekhar's equation of gravitational equation. Treating Stoner's minimum energy principle as a variational problem in which the total energy is a functional of the density, and this density is a variable function of the radial density, this variational approach leads to the quantum mechanical ground state of an electron gas in the gravitational field of the ions, which maintain charge neutrality. This connection explains why Stoner and Chandrasekhar obtained the same relations for the density and mass of the star as functions of fundamental constants, but with somewhat different dimensionless quantities. In particular, I will show that the solution to the generalized form of Stoner's variational problem for the minimum of the total energy of a dense star leads to the differential equation of gravitational equilibrium which Chandrasekhar applied in his work. I have not found any evidence, however, that either Stoner or Chandrasekhar were aware of this connection.

The total energy E of a zero temperature dense star supported entirely by degeneracy pressure against the gravitational attractive forces can be written as a functional of the mass density distribution ρ integrated over the volume of the star,

$$E = \int dv(\varepsilon(\rho) - u(\rho, r)), \tag{1}$$

where $\varepsilon(\rho)$ is the internal energy given as a function of of the mass density ρ by Stoner' relativistic equation of state for a electron degenerate gas [16], $u(\rho, r)$ is the gravitational energy $u(\rho, r) = -(1/2)G\int dv'\rho(r')\rho(r)/|r - r'|$, and G is Newton's gravity constant. The equilibrium distribution ρ as a function of position r can be determined by evaluating the minimum of E, subject to the condition that the total mass $M = \int dv\rho$ is fixed. Assuming that ρ depends only on the radial distance r from the center of the star, this variational problem leads to the differential equation for gravitational equilibrium,

$$dP/dr = -G\frac{M(r)\rho(r)}{r^2}, \tag{2}$$

where $P = \rho d\varepsilon/d\rho - \varepsilon$ is the pressure, and $M(r) = 4\pi\int drr^2\rho(r)$ is the mass inside the radius r. In the uniform density approximation, the solution of Stoner's minimum energy principle gives the relation $P = (3/20\pi)GM^2/R^4$, where P is the mean pressure, M is the mass and R is the radius of the star. Stoner's relativistic equation of state for the pressure - density relation of a degenerate electron gas was first given in the form $P = Ax^4F(x)$, where[52]

$$F(x) = \frac{1}{8x^3}[\frac{3}{x}\log(x + \sqrt{1+x^2}) + \sqrt{1+x^2}(2x^2 - 3)], \tag{3}$$

and $x = (h/mc)(3n/8\pi)^{1/3}$ (see Eqs. 18 and 19 in Stoner's paper [17]). Here n is the electron density $n = 3M/4\pi R^3 m_H \mu$, m is the electron mass, m_H is the proton mass, h is Planck's constant , c is the velocity of light , μ is the molecular weight and $A = (8\pi/3)(m^4c^5/h^3)$. Hence Stoner's analytic solution for the mass M of a a white dwarf takes the form $M = M_c(4F(x))^{3/2}$. In the limit of small density $x = 1$, $F(x) \approx x/5$, and $P = (1/20)(3/\pi)^{2/3}(h^2/m)n^{5/3}$, which corresponds to Fowler's result[7] for the pressure-density relation in the non-relativistic limit. In this limit we recover Stoner's original relation that the density n is proportional to the square of the mass M of the star, $n = (10\pi/3)(mc/h)^3(M/M_c)^2$. The maximum momentum of the electrons is $p = (mc)x$, and therefore when x is of order one or larger the effects of relativity become important, as was first pointed out by Anderson [15], and independently by Chandrasekhar [23]. In the limit of infinite density, $x \to \infty$, $F(x) \to 1/4$, which gives $P = (1/8)(3/\pi)^{1/3}n^{4/3}$, and $M = M_c$, with Stoner's critical mass expressed in terms of some of the fundamental constants of nature, $M_c = (3/16\pi)(5hc/2G)^{3/2}(1/m_H\mu)^2$. Chandrasekhar's result for the critical mass, expressed in terms of fundamental constants, corresponds to $M_c = u(\sqrt{6}/8\pi)(hc/G)^{3/2}(1/m_H\mu)^2$, where $u = 2.018...$ is a constant obtained by numerically integrating the equation of gravitational equilibrium for an $n = 3$ polytrope. It can be readily verified that the critical mass evaluated with a mass density distribution corresponding to an $n = 3$ polytrope is 20% smaller than for a uniform density distribution.

Appendix II: Eddington's Feb. 28, 1932 letter to Stoner

In light of Eddington's famous controversy with Chandrasekhar at a 1935 meeting of the Royal Astronomical Society [32 33], in which Eddignton quipped, without giving any reference to Stoner, that the relativistic equation of state for a degenerate electron gas

"...is based on a combination of relativity mechanics and non-relativity quantum theory, and I do not regard the offspring of such a union as born in lawful wedlock ..." [53]

it is remarkable that three years earlier Eddington had been in communication with Stoner about this equation of state, encouraging Stoner in his work, and even suggesting that they collaborate on an investigation of the effect of this equation on stellar structure. In a letter to Stoner on Feb. 28, 1932 (see Fig. 2), Eddington wrote:

"I have been thinking that a combination of your work and mine would make quite definite the state of the question as to upper limits to the temperature and density of a star of given mass. This is very important, e.g. in regard to theories of subatomic energy and does not seem to be as well understood by astronomers as it might be..."

Then he added that

"I suggest that it would be very useful to tabulate $f(\rho)$ [Stoner's relativistic equation for the pressure f as a function of the density ρ] or $f(\rho)/\rho^{5/3}$, others who have written on the subject seem to consider only the two extremes of ordinary [$f(\rho) \propto \rho^{5/3}$] and relativistic degeneracy [$f(\rho) \propto \rho^{4/3}$], whereas we are actually most concerned with intermediary conditions "

By "others" Eddington evidently was referring here to the work of Milne [20] and of Chandrasekhar [26] who, at the time, had been taking into account such "intermediary condiitions" by a crude interpolation scheme between two density regimes where either the non-relativistic or the extreme relativistic pressure-density relations were assumed to be applicable[54]. Eddington continued:

" While the critical mass may have some interest of its own, it does not affect the more fundamental questions. It is useless to suggest a theory of subatomic energy involving temperatures of 10^{11} degrees which might be possible for Sirius but could not possibly apply to Krueger 60.

We have been fairly generous in upper limits, so that (especially if there is abundance of hydrogen) the critical mass is probably much greater than the sun's"

Evidently, at the time Eddington's primary interest was the applications of Stoner's relativistic equation of state to find limits on the temperatures required for the production of *subatomic energy* in stars. The passage of his letter quoted here reveals that in 1932 Eddington did not have objections to Stoner's relativistic equation of state for a degenerate electron gas, which together with Stoner's minimum energy principle implied the existence of a critical mass. Moreover, he understood that the magnitude of this critical mass depended on the inverse square of the molecular weight μ , which had generally been assumed to be equal to 2.5. Hence, one can understand his remarks that for a hydrogen star, the critical mass would "probably be much greater than the sun's", because in this case $\mu = 1$, and the critical mass would be about nine times larger than the mass of the sun.

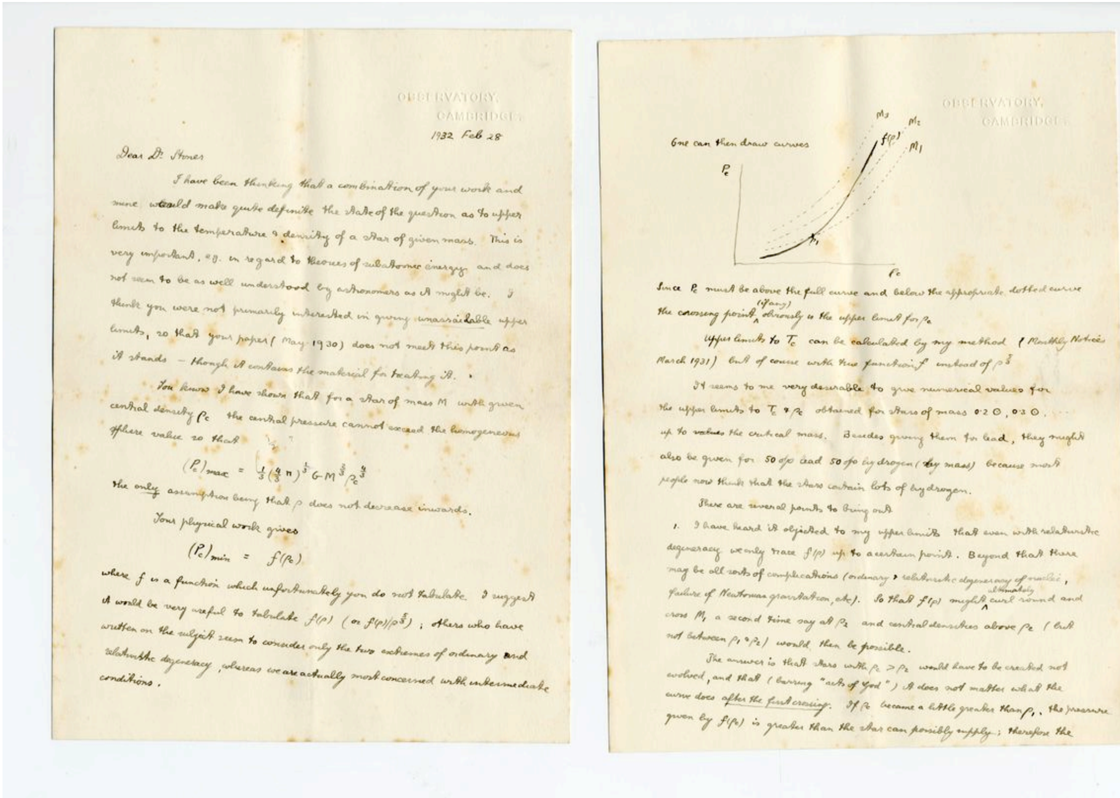
Stoner followed Eddington's suggestions by publishing additional numerical tables of his relativistic equation of state [55], and by calculating the maximum density and temperature of dense stars in the the uniform density approximation for arbitrary densities and for the polytropic density distribution in the non-relativistic and extreme relativist limits [56]. In the last of his five papers on white dwarfs , Stoner took into account the effect of radiation pressure on the equilibrium state of white dwarfs. In the introduction he reviewed his previous work:

"The question of limiting densities in connection with white dwarf stars has already been discussed in a series of papers. In the first of these (reference 10)- the relativity effect being considered in the second (reference 14) - the case of a sphere of uniform density was considered. The results may be considered as giving rough upper limits for the *mean* density. In the third paper (reference 16) the effect of non-uniform (polytropic) density distribution was discussed, some of the conclusions being similar to those reached by Chandrasekhar (reference 15) at about the same time."

Stoner had applied an inequality, which had been published earlier by Eddington[57], to obtain the maximum possible value of the density and the temperature of a star under the assumption that the central pressure was the sum of the pressure due to a degenerate electron gas and the pressure of radiation [58], finding that

"... the maximum values [of density and temperature] can be fixed by these considerations provided that the star has a mass *below a critical value* ",

namely, the mass limit which Stoner had obtained previously in the absence of radiation.



OBSERVATORY,
CAMBRIDGE.

1932 Feb 28

Dear Dr Stoner

I have been thinking that a combination of your work and mine would make quite definite the state of the question as to upper limits to the temperature & density of a star of given mass. This is very important, e.g. in regard to theories of subatomic energy and does not seem to be as well understood by astronomers as it might be. I think you were not primarily interested in giving unassailable upper limits, so that your paper (May 1930) does not meet this point as it stands – though it contains the material for treating it.

You know I have shown that for a star of mass M with given central density ρ_c the central pressure cannot exceed the homogeneous sphere value so that

$$(P_c)_{max} = \tfrac{1}{3}(\tfrac{4}{3}\pi)^{\frac{1}{3}} G M^{\frac{2}{3}} \rho_c^{\frac{4}{3}}$$

the only assumption being that ρ does not decrease inwards.

Your physical work gives

$$(P_c)_{min} = f(\rho_c)$$

where f is a function which unfortunately you do not tabulate. I suggest it would be very useful to tabulate $f(\rho)$ (or $f(\rho)/\rho^{\frac{5}{3}}$); others who have written on the subject seem to consider only the two extremes of ordinary and relativistic degeneracy, whereas we are actually most concerned with intermediate conditions.

OBSERVATORY,
CAMBRIDGE.

One can then draw curves

Since P_c must be above the full curve and below the appropriate dotted curve the crossing point (if any) obviously is the upper limit for ρ_c

Upper limits to T_c can be calculated by my method (Monthly Notices March 1931) but of course with the function f instead of $\rho^{\frac{5}{3}}$

It seems to me very desirable to give numerical values for the upper limits to T_c & ρ_c obtained for stars of mass 0·2 ☉, 0·3 ☉, ... up to the critical mass. Besides giving them for lead, they might also be given for 50 o/o lead 50 o/o hydrogen (by mass) because most people now think that the stars contain lots of hydrogen.

There are several points to bring out

1. I have heard it objected to my upper limits that even with relativistic degeneracy we only trace $f(\rho)$ up to a certain point. Beyond that there may be all sorts of complications (ordinary & relativistic degeneracy of nuclei, failure of Newtonian gravitation, etc). So that $f(\rho)$ might ultimately curl round and cross M_1 a second time say at ρ_2 and central densities above ρ_2 (but not between ρ_1 & ρ_2) would then be possible.

The answer is that stars with $\rho_c > \rho_2$ would have to be created not evolved, and that (barring "acts of God") it does not matter what the curve does after the first crossing. If ρ_c became a little greater than ρ_1, the pressure given by $f(\rho_c)$ is greater than the star can possibly supply; therefore the

Fig. 2 Feb. 28, 1932 letter from Eddington to Stoner encouraging Stoner to apply his relativistic equation of state to obtain upper limits to the density and temperature of dense stars of a given mass. In Eddington's figure the dashed curves are plots of pressure vs. density to the power $4/3$ curves for different star masses M_i, $i = 1,2,3$, which he obtained under the assumption that the ratio of radiation and gas pressure inside a star is constant [11]. The solid curve is a sketch of Stoner's relativistic pressure-density relation for a degenerate gas. (Courtesy of the Trinity College library in Cambridge, England, which holds the copyright to this letter, and the University of Leeds library, where this letter is located in the Stoner Archives).

Acknowledgements

I would like to thank Werner Israel for useful comments and information about W. Anderson, and Malcolm MacGregor for helpful editorial comments.

[1] See, for example, Chandrasekhar' s obituary by Freeman Dyson, "The death of a star ", *Nature* ccccxxxviii (2005) 1086.

[2] Edmund Clifton Stoner, 1899-1968; L. F. Bates *Biographical Memoirs of Fellows of the Royal Society* xv (Nov., 1969) 201-237.

[3] G. Cantor, " The making of a British theoretical physicist - E. C. Stoner's early career''. *British Journal for the History of Science* xxvii (1994) 277-290.

[4] The first monograph containing the new quantum theory of magnetism was E.C. Stoner, *Magnetism* (Methuen, London 1930).

[5] E. C. Stoner, ``The Distribution of Electrons among Atomic Levels'',*Philosophical Magazine xlvii* (1924) 719-736.

[6] For a historical description of the origin of the exclusion principle and Stoner's role in its formulation see, J.L. Heilbron, "The origins of the exclusion principle'', Historical Studies in the Physical Sciences xiii (1982) 261-310. Heilbron's perceptive comment, "of psychological interest is Pauli's continual misstatement of the key observations he took from Stoner'', strikes a chord, because also Chandrasekhar continual neglect to acknowledge Stoner's priority in the discovery of the unusual properties of white dwarfs is the main reasons why Stoner's contribution in this field have been forgotten .

[7] R.H. Fowler, "On Dense Matter'', *Monthly Notices of the Royal Astronomical Society*,lxxxvii (1926) 114-122. Actually, Fowler, and later also Chandrasekhar, refered to the degeneracy pressure of electrons as due to Fermi-Dirac statistics, which is based on Pauli's exclusion principle But all subsequent white dwarf calculations were done at zero temperature, and in this case quantum statistics does not play any role, and only the exclusion principle is required.

[8] Stoner's work, ref. 5 , which led to Pauli's formulation of the exclusion principle, indicates that R. H. Fowler's first encounter with this principle occurred through his contact with Stoner. In ref. 2 , p. 214, Stoner recollects that

"One night in May 1924, a distribution scheme occurred to me in which the numbers in full levels were simply related to the quantum numbers specifying them, and which seemed free from the (usual admittedly) arbitrary and unsatisfactory features in schemes previously proposed. I was very excited about this, and in the next few days I satisfied myself that it was consistent with the major relevant experimental findings. I wrote a brief not about the scheme for Rutherford and, in his absence. left it on his desk. He must have passed it on to R.

H. Fowler (with whom, at this period, I had several most helpful discussions on theoretical points), for soon afterwards Fowler asked me to call on him to discuss it. He was favourably impressed, and suggested that I should write a full and detailed paper about it. This I was only too pleased to do, and in July a paper on `The distribution of electroms among atomic levels' was completed. It was communicated by Fowler to the *Philosophical Magazine,* and appeared in the issue of October 1924."

[9] Actually, the density of the companion of Sirius, one of the only three known white dwarfs which was known at the time, was underestimated by an order of magnitude.

[10] For the history of the early observation of a white dwarf, see
J.B Holberg and F. Wesemael, ``The discovery of the companion of Sirius and its aftermath'', *Journal for the History of Astronomy* xxxviii (2007) 162-174.

[11] A.S. Eddington, *The Internal Constitution of the Stars*, First edition 1926, (Dover Publication 1959) pp. 82-83.

[12] L. H. Thomas, "The Calculation of Atomic Fields", *Proceedings of the Cambridge Philosophical Society* xxiii (1927) 542-548. Thomas had been a student at Trinity College, Cambridge, where Fowler had been appointed a College lecturer in Mathematics in 1920, but at the time Thomas wrote this paper he was visiting Bohr in Copenhagen. In the case of an atom, the forces are electrostatic which are repulsive between electrons and attractive between an electron and the nucleus. Treating the electrons as a degenerate gas, Thomas arrived at an equation equivalent to the Lane-Emde equation for a polytrope of index $n = 3/2$, but with an opposite sign. A year later Enrico Fermi also derived the same equation which is now known as the Thomas-Fermi equation, E. Fermi, "Über die anwendung der statistischen methode auf die probleme des atombaues", *Falkenhagen,Quantentheorie un Chemie, Leipziger Votraeger* (1928) 95-111. The application of the $n = 3/2$ polytrope to obtain the properties of white dwarfs for the non-relativistic equation of state for degenerate electrons was first carried out by E. A. Milne (ref. 20), who referenced Thomas' work, and a year later by Chandrasekhar (ref. 18).

[13] E. C. Stoner, "The Limiting Density in White Dwarfs'', *Philosophical Magazine* vii (1929) 63-70.

[14] About 35 years ago, without being aware of Stoner's seminal work, I applied the same energy minimum principle to obtain the properties of white dwarfs in the uniform density approximation, with an approximate form of the relativistic equation of state similar to Stoner's, (see M. Nauenberg, ``Analytic

approximations for the mass-radius relation and energy of zero-temperature stars'', *The Astrophysical Journal* clxxv (1972) 417-430). My results were similar to Stoner's (compare the mass-radius relation for a white dwarf shown in Fig. 1 of this ApJ journal , with Fig. 1 in this paper obtained from Stoner's analytic result). At the time, I sent a pre-print of my article to Chandrasekhar with a cover letter asking for his comments, but unfortunatley I did not receive a response which could have alerted me to Stoner's work.

[15] W. Anderson, `` Über die Grenzdichte der Materie und der Energie'', *Zeitschrift fur Physik* liv (1929) 851-856.

[16] This relativistic equation of state for a degenerate electron gas is often called the Anderson-Stoner equation. But Anderson' s relativistic analysis and his formulation of this equation , given in ref.15, is incorrect.

[17] Stoner, E.C. ``The Equilibrium of White Dwarfs'', *Philosophical Magazine* ix (1930) 944-963.

[18] S. Chandrasekhar, " The Density of White Dwarfs", Philosophical Magazine xi (1931) 592-597.

[19] E. C. Stoner and F. Tyler, " A Note on Condensed Stars" , Philosophical Magazine xi (1931) 986-993. In this paper the authors did not obtained the energy minimum for the extreme relativistic equation state by taking density distribution for the $n = 3$ polytrope . This calculation leads to the same value for the critical mass obtained by Chandrasekhar, although in this limiting case the energy minimum vanishes. Stoner's condition, that the derivative of the energy with respect to the central density is zero (see Appendix I), is satisfied because the energy itself also vanishes in this limit. These subtle mathematical issues may have been the reason why the authors did not attempt to do this calculation.

[20] E. A. Milne, ``The Analysis of Stellar Structure'' , *Monthly Notices of the Royal Astronomical Society* xci (1930) 4-55.

[21] S. Chandrasekhar, ``The Ritchmyer Memorial Lecture- Some Historical Notes'', *American Journal of Physics* xxxvii (1969) 577-584.

In his account, Chandrasekhar recalls that,

"Soon after arriving in England, I showed these results to R. H. Fowler. Fowler drew my attention to two papers by Stoner, one of which had appeared earlier that summer. In these two papers Stoner had considered the energetics of homogeneous spheres on the assumption that the Fermi-Dirac statistics prevailed in them. While Stoner' s result gave some valid inequalities for the problem, he had not derived the structure of the equilibrium configurations in which all the governing equations are satisfied. Fowler, of course, appreciated this difference, and he was satisfied with detailed results pertaining to the nonrelativistic configurations. But he appeared skeptical of my result on the critical mass, and so was E. A. Milne to whom he communicated it "

[22] Like Anderson, Chandrasekhar arrived at this conclusion by applying the same relation between mean momentum and the cube of the density of a degenerate electron gas in conjunction with the non-relativistic mass-density relation for a white dwarf (see Appendix I).

[23] S. Chandrasekhar, ``The Maximum Mass of Ideal White Dwarfs'', *Astrophysical Journal* lxxiv (1931) 81-82.

[24] 1977 interview with Spencer Weart (Niels Bohr Library, American Institute of Physics)

[25] Fowler forwarded Chandrasekhar's result to Edward A. Milne, who was an astrophysicist at Oxford University. Milne, " while acknowledging that Chandrasekhar had worked out the relativistic degenerate star "most beautifully" , wrote to him that " the flaw in your reasoning is that you cannot prove that the solution appropriate to the outer parts of the relativistic degenerate core is Emden's solution, it may be one of the others", quoted in ref. 32 , p. 121. For any polytropic solution, the density decreases uniformly from the center of the star and vanishes at its boundary. Hence, for a sufficiently large central density the extreme relativistic equation of state is a valid approximation in the core, but it would fail near the boundary where the electrons become non-relativistic. Eventually, Chandrasekar was led to the conclusion that a consistent solution for the critical mass required that the central density become infinite, as Stoner had shown earlier in the uniform density approximation, because in this case the envelope, where the electrons would be non-relativistic, vanishes.

[26] S. Chandrasekhar, `` The Highly Collapsed Configurations of a Stellar Mass'', *Monthly Notices of the Royal Astronomical Society* xci (1931) 456-466.

[27] S. Chandrasekhar, `` The Highly Collapsed Configurations of a Stellar Mass (second paper)'', *Monthly Notices of the Royal Astronomical Society* xcv (1935) 207-225.

Chandrasekhar's numerical results are given in Table I of his paper which is reproduced as Table 25 in his book *An introduction to the study of stellar structure* (University of Chicago Press, 1939). The curve given here in Fig. 1, which is based on Stoner's 1930 *analytic* calculation (see reference 17), is nearly identical to Fig. 2 in Chandrasekhar's paper, reproduced as Fig. 31 in his book.

[28]E.C. Stoner, `` The Minimum pressure of a Degenerate Electron Gas'', *Monthly Notices of the Royal Astronomical Society* xcii (1932) 651-661.

[29] E.C. Stoner, `` Upper Limits for Densities and Temperatures in Stars'', *Monthly Notices of the Royal Astronomical Society* xcii (1932) 662-676.

[30] In the last of his five papers on white dwarfs, Stoner followed Eddington's suggestion to apply his relativistic equation of state for a degenerate electron gas taking into account the effect of radiation pressure on the equilibrium state of dense stars.

[31] For a complete list of Chandrasekhar's papers on white dwarfs, see his Nobel lecture, S. Chandrasekhar, `` On Stars, their evolution and their stability'' *Nobel Lectures in Physics 1981-1990* (World Scientific, Singapore,1995) 142-164.

[32] K.C. Wali, *Chandra, A Biography of S. Chandrasekhar* (University of Chicago Press 1991)

[33] A. Miller, *Empire of the Stars, Obsession, Friendship, and Betrayal in the Quest for Black Holes* (Houghton Mifflin, Boston 2005)

[34] At a meeting of an American Physical Society meeting at Stanford University in December 1933, Walter Baade and Fritz Zwicky made the suggestion that the origin of supernova explosions was due to the collapse of massive stars into neutron stars. But apparently they were unaware of the existence of a white dwarf mass limit, because this limit was not mentioned in the abstract to their report which appeared as a letter to the editor , "Remarks on Super-Novae and Cosmic Rays", in *Physical Review* xlvi (1934) 76-77. The first to make this connection was George Gamow who referred only to Chandrasekhar's work on the mass limit (see G. Gamow, "Physical possibilities of stellar evolution", *Physical Review* lv (1939) 719-72). At a 1939 conference on astrophysics at the College the France, Chandrasekhar also recognized that "the origin of Supernova phenomena" may be due to the collapse of stars more massive than the white dwarf critical mass; see Chandrasekhar, " The white dwarfs and their importance for theories of stellar evolution", reprinted in *A Quest for Perspectives: Selected works of S. Chandrasekhar*, edited by K. C. Wali (Imperial College Press London 2001) p. 104.

[35] In 1939 Chandrasekhar met Eddington at high table dinner in Cambridge and

asked him: " How much of your fundamental theory depends on your ideas on relativistic degeneracy? ". Eddington replied, " Why, all of it ", ref. 33 p. 131.

[36] A. S. Eddington, "On Relativistic Degeneracy", *Monthly Notices of the Royal Astronomical Society* xcv (1935) 194-206;

"Note on relativistic degeneracy" *Monthly Notices of the Royal Astronomical Society* xcvi (1935) 20-21;

"The pressure of a degenerate electron gas and related problems", *Proceedings of the Royal Society* A clii (1935) 253-272.

Eddington's re-examination of relativistic degeneracy led him to some extreme statements such as: " The Stoner-Anderson modification is fallacious … a rigorous treatment leads to the original [Fowler] equation of state", and " The Stoner-Anderson formula does not exist", quoted in L. Mestel, "Arthur Stanley Eddington: pioneer of stellar structure theory" *Journal of Astronomical History and Heritage* vii (2004) 65-73

[37] A detailed critique of Eddington's objections to Stoner's relativistic equation of state for a degenerate electron gas can be found in E. Schatzman, ``White Dwarfs'' (North-Holland 1958) pp. 68-63.

[38] Pauli, whose opinion was also requested, responded sarcastically that " Eddington did not understand physics", quoted in ref. 32 , p. 131.

[39] On January 23, 1931, replying to the umpteen letter from Chandrasekhar, Leon Rosenfeld wrote with respect to Eddigton's objection to relativistic degenerary, " Wouldn't it be a good policy to leave him alone, instead of losing one's time and temper in fruitless arguments?.." , quoted in ref. 32, p. 131.

[40]C. MØller and S. Chandrasekhar, "Relativistic Degeneracy", *Monthly Notices of the Royal Astronomical Society* xcv (1935) 673-676.

[41] R. Peierls, ``Note on the derivation of the equation of state for a degenerate relativistic gas'', *Monthly Notices of the Royal Astronomical Society* xcvi (1936) 780-784.

[42] J. McDougall and E.C. Stoner, "Computation of Fermi-Dirac Functions", *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* ccxxxvii (1938) 67-104. During a discussion period after his presentation at the 1939 conference on Astrophysics in Paris , Chandrasekhar also mentioned that "one of his pupils and Dr. Stoner are working on the problem of what happens in the region in which the equation of state of degenerate matter approaches the equation of state of a perfect gas" (ref. 34 p.170).

[43] T.D. Lee, "Hydrogen content and energy-productivity mechanism of white dwarfs", Astrophysical Journal cxi (1950) 625L

[44] L. Landau, "On the theory of Stars*", Physikalische Zeitschrift der Sowjetunion* i (1932) 285-288. At the time, Landau was visiting the ETH in Zurich where Perierls was an *assistent* of Pauli. During his visit he wrote a fundamental paper on relativisitic quantum field theory with Peierls, and developed the quantum theory of diamagnetism associated with a degenerate electron gas in a metal. Thus, it was natural that he should consider also the role of relativisitic degeneracy of an electron gas in the interior of stars. Landau's paper was submitted for publication on February 1931, *before* the appearance of Chandrasekhar's paper [23], which had been submitted for publication only three months earlier, and appeared in the March 1931 issue of the Astrophysical Journal [23]. But in " some historical notes" [21], Chandrasekhar stated that " Landau isolated the critical mass apparently without knowledge of my results published *two years earlier* [my italics]", and in his reference to Landau's article he dated its publication to 1933, although it had appeared a year earlier. In his biography "Chandra", K. Wali corrected this error, ref. 32 p. 121, but then he stated that Landau's paper was published " a year later" than Chandrasekhar's, without clarifying that the additional nine month delay was due to the slower publication rate of the Soviet journals where Landau's paper appeared. Like Stoner, Landau also recognized that the equilibrium state of dense stars should be a minimum of the energy, and then he noticed that for the extreme relativistic equation of state, a star of fixed mass would have to either expand or "collapse to a point" to attain this minimum . This led him to evaluate the critical mass which separated these two regimes by "solving the $n = 3$ polytropic equation of Emde" . During a later visit with Bohr in Copenhagen, on the same day that the news of Chadwick's discovery of the neutron appeared, Landau suggested the possibility of formation of neutron stars, where the source of internal pressure is now due to degenerate neutrons rather than to electrons (ref. 46 p. 224) . But he did not publish his idea until five years later, L. Landau, "The origin of Stellar Energy", *Nature* cxxxxi, 333-334.

[45] S. Chandrasekar, "An Introduction to the study of Stellar Structure" , (Dover, New York, 1967),

[46] W. Israel, *Dark Stars: the evolution of an idea* in ``300 Years of Gravity'' edited by S. Hawking and W. Israel (Cambridge Univ. Press, Cambridge 1987) 199-276; W. Israel, ``Imploding Stars, Shifting Continents, and the Inconstancy of Matter'' , *Foundations of Physics* xxvi (1996) 595-626.

[47] *A source book in astronomy and astrophysics* 1900-1975, edited by K.R. Lang and O. Gingerich (Harvard Univ. Press, Cambridge, 1979) pp. 433-444.

[48] Fred Hoyle credited Stoner with the discovery of the white dwarf mass limit (Wali, private communication). His former student, Leon Mestel, also mentioned Stoner and Anderson in connection with this limit, see L. Mestel, "The theory of white dwarfs", *Monthly Notices of the Royal Astronomical Society* cxii , (1952) 583-597.

[49] S. Chandrasekhar, Eddington, the most distinguish astrophysicist of his time (Cambridge 1983).

[50] S. Chandrasekhar, ``The Highly Collapsed Configurations of a Stellar Mass'', *Monthly Notices of the Royal Astronomical Society* xci, (1931) 456-466.

[51] S. Chandrasekhar, ``Some Remarks on the State of Matter in the Interior of Stars'', *Zeitschrift fur Astrophysik* v (1932) 321-326.

[52] In ref. 27, Stoner introduced the notation $P(\rho) = A' f_P(x)$, where $f_P(x) = x^4 F(x)$. In his 1935 paper (ref. 26) , and in his book (ref. 39), Chandrasekhar defines $P(\rho) = A_2 f(x)$, where $f(x) = 8 f_P(x)$, and $A_2 = A/8$

[53] "Discussion of papers 4. and 5 by A. S. Eddington and E. A. Milne", *The Observatory* lviii (1935) 37-39.

[54] As late as 1934, Chandrasekhar still thought that the only "possible equations of state" for a degenerate electron gas were either the non-relativistic or the extreme relativistic forms of Stoner's exact equation of state, see S. Chandrasekhar, "The physical state of matter in the interior of stars", *The Observatory* lvii (1934) 93-99

[55] E.C. Stoner, "The Minimum pressure of a Degenerate Electron Gas'',*Monthly Notices of the Royal Astronomical Society* xcii (1932) 651-661.

[56] E. C. Stoner, "Upper Limits for Densities and Temperatures in Stars'',*Monthly Notices of the Royal Astronomical Society* xcii (1932) 662-676.

[57] A. S. Eddington, "Upper limits to the Central Temperature and Density of a Star'', *Monthly Notices of the Royal Astronomical Society*, xci (1931) 44-446

[58] For a given star mass M and central density ρ_c, Eddington had shown that the central pressure P_c must be *less* than the central pressure of a star with *uniform* density ρ_c, see ref. 55. Eddington's theorem is the inequality

$P_c < (1/2)(4\pi/3)^{1/2} GM^{2/3} \rho_c^{4/3}$, which is mentioned in his letter to Stoner (see Fig. 2), but with the first factor on the right hand side of this inequality given incorrectly as $(1/3)$.