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Stellar Distribution and The Law of Chance

Second paper:

Probabilities of Geometrical Configurations, and the Irregularity of Stellar Distribution in the Paris Carte-du-Ciel Zone $\delta = +24^0$

By

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K. Mattiesen'i trükikoda o.-ü., Tartu 1934.

I. Introduction and Summary.

The present paper, which has been prepared with the assistance of the computing staff of Tartu observatory, represents the continuation of an earlier publication [1]; the method of calculating the probability of a positive excess in the frequency of stellar densities on a chart is made here more precise and practicable; also, for geometrical configurations of adjacent areas, numerical values of the probabilities are calculated, whereas in the first paper these configurations were judged merely "by impression". A rediscussion from the new standpoint of the irregularity of stellar distribution in the Paris Zone is made, and some conclusions, arrived at in the first paper, are subject to revision.

The mathematical analysis of stellar distribution enables us to locate irregularities of stellar distribution with a certain calculable degree of probability, in many cases approaching certainty. But with respect to the physical cause of the observed irregularities, the statistical analysis of a single photograph does not give us positive information; we cannot tell whether an observed vacancy is due to the presence of an obscuring cloud, or whether it represents a real vacancy, a region of physical avoidance. A positive answer to the question may be got from photographs of the same region showing stars to fainter limiting magnitudes, where the effect of an obscuring veil must persist whereas the effect of stellar clustering will be changed. In this way Shapley [2] has shown that many of our "obscured" regions published in the first paper are most probably due to real irregularities in the distribution of stars in space, and that in some other cases ab-

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1) "Stellar Distribution etc, with a special discussion of the Paris Carte-du-Ciel Zone $\delta = +24^0$", By E. Öpik and Miss M. Lukk, T. P. **26.** 2, 1924. Referred to in the following as the first paper.

2) H. Shapley, Note on Obscuring Clouds in High Galactic Latitudes, Harvard Circular **281,** 1925.

sorption by dark matter appears probable. We may add that most of the non-confirmed regions of obscuration do not show unusual geometrical configurations, in the light of the present analysis. Our analysis of geometrical configurations indicates that among the fifteen Paris charts in galactic latitude over 23°, and for which star counts were repeated at Harvard, only 4.2 or 28 per cent (mathematical expectation) have true vacancies, whereas the rest of the vacancies are by no means unusual and must be regarded as due to pure chance. The Harvard counts indicate that these few true vacancies in high galactic latitudes are not due to absorption of light by cosmic clouds.

A positive result of our statistical analysis is always ambiguous with respect to its physical interpretation; but a negative result is definite, at least when it refers to a statistical entity, and not to an individual case. Of the filty Paris charts between galactic latitude 45° and 87°, only one, Nr. 73 ($B = + 46^0$), shows a somewhat unusual vacancy, the probability of its reality being 0.94; none of these charts do show any unusual excess of low star densities; whereas for 27 per cent of these charts the distribution of high densities and their geometrical configurations are unusual, or "real". This may be interpreted as a practical absence of obscuring clouds, and a moderate but conspicuous effect of real irregularity in stellar distribution ("clustering") in the part of the zone with galactic latitude over 45°. For the rest of the Paris zone, in lower galactic latitudes, a marked preponderance of "clustering" persists, the irregularities on the high density side being more frequent than among low star densities; the unequality, however, is not so conspicuous as for the high galactic latitudes; it appears that also in low galactic latitudes clustering is more prominent than obscuration; but the existence of some confirmed and well known regions of obscuration indicates that clustering cannot be regarded as solely responsible for apparent irregularities in stellar distribution. We remind that these conclusions refer to an average distance of stars of the fourteenth magnitude.

In our first paper, a valuable research by Pahlen [1]), referring to the same subject, was overlooked; Pahlen's paper is apparently

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[1]) E. von der Pahlen, Über die Wahrscheinlichkeit von Sternverteilungen. Dissertation. Göttingen 1909.

the first attempt of an investigation of stellar distributions from the standpoint of the law of chance. Unfortunately, his method of attacking the problem is not only rather complicated, but also of little use in real cases; his method of calculating the probability of a whole observed distribution, being mathematically correct, depends too much upon different natural and observational sources of error; e. g. he gets a probability of 10^{-21} for a chart[1]) which apparently shows but a very slight deviation from the theoretical distribution; by the method of calculation exposed below, we find that the positive excess on the low density side has a probability 0.09, whereas on the high density slope the probability is 0.08, values which are not unusual and by no means comparable with Pahlen's extraordinary value. The explanation is, that Pahlen's method of calculating the probability of a given distribution depends upon all possible factors influencing the distribution, among which a number of irrelevant factors such as errors of observation, non-uniformity of the telescopic field etc[2]), are always present; if considered from such a standpoint, an observed distribution is always improbable, and an over-sensitive method such as Pahlen's may give low probabilities in cases where the deviations from a chance distribution are irrelevant. The method of calculating the probabilities given below is insensitive, especially to small influences; also, the method is applied not to a whole distribution, but only to its two extreme portions, where the influence of real factors, such as obscuration or clustering, is felt the most acutely. It seems that in this manner we have succeeded in calculating real probabilities, referring to deviations from more or less a chance distribution, such which may appear as the result of different observational and natural irrelevant influences; these probabilities may serve as a measure of the reality of relevant factors.

II. Probability of a Positive Excess.

With respect to the general statement of the problem and to notations, we refer to the first paper. We are concerned only with a certain kind of deviations from a chance distribution, such

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1) Loc. cit., p. 22, table for $N = 1163$, $n = 169$.

2) Compare T. P. **26.** 2, pp. 8—21.

which may be the result of a cosmical cause; such deviations must have always the character of a positive excess; in the distribution of star densities over a given region, a positive excess means that the frequency of small and large densities is greater than expected from the law of chance; intermediate densities are deficient in this case, as a natural consequence of the excess on the extremities of the frequency table. We might treat this deficiency of intermediate densities, but it does not seem profitable; first, because the "central defect" represents often only a small percentage of the frequency itself and is too much subject to accidental variations; secondly, because the two extreme branches of the frequency table are to some extent independent and permit of a separate treatment of high and low densities. Thus, we have to limit our attention to the extremities of the frequency table.

In the first paper, formula (11)[1] gave an expression for the combined weight, or the reciprocal of the mathematical expectation per chart, of a positive excess; this formula, being based on the theorem of multiplication of probabilities, is somewhat sensitive to irrelevant influences, although not in such a degree as Pahlen's formula; as may be seen from sample computations[2], our formula gave rather large weights produced merely by observational errors (plus influence of double stars). For $N = 800$, these spurious weights were 5 and 32 for the two branches respectively; for $N = 1000$, the weights were 45 and 128. Thus, not only the spurious weights are large, but they are extremely sensitive to the total number of stars per chart. The method described below is practically free from this inconvenience.

The method is explained the most conveniently with the aid of the following example referring to Paris Chart 168.[3]

Instead of using, as in the first paper, the individual frequencies, we take the cumulative frequencies $n'_0 = \Sigma\, n_0$ (observed), and $n'_c = \Sigma\, n_c$ (theoretical), and calculate the cumulative excess, $n'_0 - n'_c$; for the sake of definiteness, we limit our attention only to the part of the table for which the cumulative theoretical frequency does not exceed 40, which is

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[1] T. P. **26**. 2, p. 22.

[2] Ibidem, p. 23–24.

[3] T. P. **26**. 2, p. 169.

Paris Chart 168.

Computation of the Weight of a Positive Excess.

Low density branch of the distribution of densities

Star density, r	0	1	2	3	4	5
Observed frequency, n_o	1	0	2	12	16	20
Theoretical frequency, n_c	0.2	0.9	3.1	7.4	13.6	19.8
$n'_o = \Sigma n_o$	1	1	3	15	31	51
$n'_c = \Sigma n_c$	0.2	1.1	4.2	11.6	25.2	45.0
Excess, $n'_o - n'_c$	$+0.8$	-0.1	-1.2	$+3.4$	$+5.8$	...
c	1.3	...	...	5.3	7.6	...
$x = \frac{n'_o - n'_c}{c}$	0.6	...	...	0.6	0.8	...
W	5	...	...	5	8	...

slightly smaller than one-quarter of the total sum of frequencies, 169; thus we do not take into account the data referring to $r = 5$, because $n'_c = 45.0 > 40$. In the table we find three positive, and two negative excesses; for the positive excesses, according to the approximate method described in the first paper[1]), we calculate the individual weights, W. These individual weights represent inverse probabilities for the given individual frequencies to have a positive excess, equal or greater than the observed value.

Of these weights, we choose the greatest, in the present case $W = 8$. This figure, however, cannot be used without a correcting factor. It represents a selected value, out of a number of individual cases, or individual experiments — five in the example chosen; chances to find large deviations in this way are thus greater than chances in a single experiment. On the other hand, the five individual cases are not equivalent to five independent experiments, because each cumulative sum contains the preceding sum also. The independent fraction of the cumulative frequency is evidently $\frac{n_c}{n'_c}$ (only theoretical frequencies are to be considered here), and the effective number of independent experiments is given by

$$s = \Sigma \frac{n_c}{n'_c} \quad \ldots (1),$$ the first member of the sum, however, being always put equal to 1. In the following, the sums (1) were computed only for $n'_c > \frac{1}{2}$. This restriction

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[1]) Loc. cit., pp 22—23.

represents a practical simplification of a more complicated theoretical requirement. In our sample, we find thus $s = 1 + 0.7 + 0.6 + 0.6 = 2 \cdot 9$.

The final weight of the positive excess in a given branch of a frequency-function may be defined now as

$$W_1 = \frac{W \text{ max.}}{s} \ldots (2);$$

we shall use the notation W_1 for the weight of the ascending, or the low density branch, and W_2 — for the weight of the descending, or the high density branch. These weights we may use not only for the computation of the frequency of occurrence of positive excesses in chance distributions, but also as an indirect measure of the strength of a given excess. We will call the stronger one the excess with the greater weight.

This final weight, W_1 or W_2, is not an inverse probability any more; it evidently represents the inverse mathematical expectation (per chart) of an excess equal or stronger than the observed one; W_1 is the reciprocal of the cumulative mathematical expectation of the weight itself. But, for large weights, mathematical expectation becomes practically identical with probability. The accurate expression for the probability to find at least one, or more excesses equal to W_1, or stronger, would be $\pi = 1 - (1 - \frac{1}{sW_1})^s$. The use of this formula, however, is inconvenient because of s being variable from chart to chart. We prefer to make use of Poisson's approximation for the probability π_n of an observed frequency n, when the theoretical frequency n_c is known:

$$\pi_n = \frac{n_c^n}{n!} e^{-n_c} \ldots (3)^{1)} .$$

In the present case, $n_c = \frac{1}{W_1}$; the probability of an excess weaker than W we get by putting in (3) $n = 0$; thus $\pi_0 = e^{-\frac{1}{W_1}}$.

Hence the probability for the excess to be equal or greater than W_1 is $\pi = 1 - e^{-\frac{1}{W_1}} \ldots (4)$.

The error introduced by this formula is small; in the following we find that important values of W exceed 2, and that s is

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1) Compare also T. P. **26.** 2, pp 6—7 and formula (1′).

mostly close to 3; substituting $W_1 = 2$, $s = 3$ in the exact formula, we get $\pi = 0.421$, whereas (4) gives $\pi = 0.393$. For greater values of W_1, the approximation is much closer.

The "spurious" weights, due to different sources of error, come out as follows[1]:

Number of stars per chart, N		300	1000
Average spurious weights, first paper	W_1	5	45
	W_2	32	128
Spurious weights, present method	W_1	2.7	1.7
	W_2	1.4	1.1

The spurious weights, computed according to the present method, are small and decrease with increasing number of stars; this latter circumstance is especially favourable for the statistical discussion. It seems also that, as the result perhaps of some systematical error of counting[2], the "spurious" positive excess is practically zero. In any case, subsequent statistics of the weights[3] indicates that the spurious weights need not be taken into account.

III. Probabilities of Geometrical Configurations.

Let us consider a chart subdivided into ν squares ($\nu = 169$ considered here); let a number m of these squares be selected by some criterion, independent of the position on the chart; in a chance distribution, the star density, or number of stars in the square, may be used as such a criterion. For real purposes, it is natural to choose the squares either by their low density — from zero to a certain upper limit; or by their high density, exceeding a certain limit. The problem is to find the probability for a certain number out of these m squares to form an adjacent group; or, to find the average number of single, double, triple etc groups formed by these m squares in a chance distribution.

As adjacent we count two squares having one side in common; those which are placed in a diagonal direction, having

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[1] Compare T. P. **26**. 2, pp. 23—24.

[2] Ibidem, p. 25.

[3] Compare Table IV of the present paper.

only one corner point in common, we do not regard as adjacent. Fig. 1 may serve as an illustration, representing actual configurations on a Paris chart. The thirty four squares of smallest star numbers show following groups: two groups of eight squares each; three groups of two; twelve single squares. The configuration is rather unusual, the weight as given below being of the order of 10^5 (computed as explained below). The thirty nine squares of highest star density are grouped as follows: one group of eight squares; one group of seven; one group of six; one group of five; one group of two; eleven single squares. The weight of the configuration is 70.

	−			−	+			+				+
−			+		+			−		+		
−				+			−		−			
−		+							−			+
−		+		−	+	−	−	−		+		
−						+	−			−		
		−		−		−	−		−	−	−	−
		−	−	−	−			−		−		−
	−		+		−	−		+	+		−	
+	+	−										
+								−		+		+
+							+	+	+		+	
+	+	+	+				−		+	+	+	+

Fig. 1.
Paris Chart Nr. 127.
+ .... densities below 5
— .... densities above 9.

The exact theory of the frequency of different adjacent groups is extremely complicated, especially because not only the number of squares in a group, but also its form, the arrangement of the single elements influences the probability. We chose here the more convenient way of approximate theory checked by direct experiment. We disregard the geometrical arrangement in a group, and consider only the number of elements in it as determining the probability. Relying upon the check by experiment, and for the sake of brevity, we do not give the derivation of our formulae; only the final formulae are given as follows. Let $x_{1,m}$; $x_{2,m}$; $x_{3,m}\ldots$ denote the average frequency of single, double, triple, etc groups formed by a total number m of selected squares. When the values of x are known for a certain m, the corresponding values for $m+1$ are given by the following approximate formulae, referring to $\nu = 169$:

$$\left.\begin{aligned}
x_{1,m+1} &= x_{1,m} + \frac{169-m-a_m}{169-m} - \frac{3.7\,x_{1,m}}{169-m} \\
x_{2,m+1} &= x_{2,m} + \frac{3.7\,x_{1,m}}{169-m} - \frac{5.7\,x_{2,m}}{169-m} \\
x_{3,m+1} &= x_{3,m} + \frac{5.7\,x_{2,m}}{169-m} - \frac{7.7\,x_{3,m}}{169-m} \\
\cdots\cdots &\quad \cdots\cdots\cdots \quad \cdots\cdots \\
\text{where } a_m &= 2m+1.7\left(x_{1,m}+x_{2,m}+x_{3,m}+\ldots+x_{m,m}\right)
\end{aligned}\right\}\cdots\cdots(5)$$

The computation starts with $m = 1$, $x_1 = 1$, $x_2 = x_3 = \ldots = 0$.

For another total number of squares, the numerical coefficients are to be changed; e. g., for $\nu = 100$, we have to substitute for 3.7, 5.7, 6.7, 1.7 etc the figures 3.6, 5.6, 6.6, 1.6 etc, and for 169 put 100. The formulae are the more accurate, the smaller the ratio $\frac{m}{\nu}$ is; it is not advisable to use the formulae when this ratio sensibly exceeds 0.25.

The experiments were made with 169 cards, each representing thus a certain square of the chart; in each series of experiments, a certain number m of the cards were specially marked. The cards were carefully mixed and got thus arranged into an accidental sequence. Assuming the ordinal number of a card in this sequence to represent one definite square on the chart, an artificial chart with a presumably random distribution of the m marked squares could be constructed. The results of the experiments, as compared with the approximate theory, are contained in Table I.

The agreement of theory and experiment is very good up to $m = 20$; above that, certain systematic divergences occur. Combining theory and experiment, and using graphical methods of interpolation, the final Table II was constructed; in this table, the frequencies of x_2 to x_5 are chiefly from experiment, and were interpolated for intermediate values of m, whereas the frequencies of higher groups are from theory and are calculated individually; $x_2, x_3, \ldots$ being given, x_1 was computed from the condition $x_1 + 2x_2 + 3x_3 + \ldots + mx_m = m$.

Table I.

Experiments with Random Arrangements of m Selected Squares among a total of 169 squares

a) $m = 5$
264 experiments

Group		x_1	x_2	x_3	$\geqslant x_4$
Total observed		1197	60	1	0
Average per chart	obs.	4.53	0.23	0.004	0
	theor.	4.58	0.21	0.007	2.10^{-4}

b) $m = 10$
128 experiments

Group		x_1	x_2	x_3	$\geqslant x_4$
Total observed		1030	110	10	0
Average per chart	obs.	8.05	0.86	0.08	0
	theor.	8.12	0.83	0.074	0.007

c) $m = 15$
88 experiments

Group		x_1	x_2	x_3	x_4	$\geqslant x_5$
Total observed		957	131	23	8	0
Average per chart	obs.	10.88	1.49	0.26	0.09	0
	theor.	10.7	1.69	0.26	0.035	0.005

d) $m = 20$
64 experiments

Group		x_1	x_2	x_3	x_4	x_5	$\geqslant x_6$
Total observed		832	162	30	6	2	0
Average per chart	obs.	13.00	2.53	0.47	0.09	0.03	0
	theor.	12.4	2.70	0.55	0.11	0.021	0.004

e) $m = 25$
48 experiments

Group		x_1	x_2	x_3	x_4	x_5	x_6	$\geqslant x_7$
Total observed		703	149	35	12	8	1	0
Average per chart	obs.	14.65	3.10	0.73	0.25	0.17	0.02	0
	theor.	13.3	3.70	0.96	0.24	0.06	0.013	0.004

f) $m = 30$
40 experiments

Group		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\geqslant x_8$
Total observed		596	156	57	16	6	1	3	0
Average per chart	obs.	14.90	3.90	1.42	0.40	0.15	0.02	0.08	0
	theor.	13.5	4.64	1.46	0.44	0.13	0.038	0.010	0.004

g) $m = 35$
32 experiments

Group		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\geqslant x_8$
Total observed		503	136	59	23	10	2	2	0
Average per chart	obs.	15.72	4.25	1.84	0.72	0.31	0.06	0.06	0
	theor.	13.2	5.41	2.02	0.72	0.25	0.085	0.029	0.012

Table I. Continued.

f) m = 40
32 experiments

Group	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\geqslant x_9$
Total observed	462	140	70	28	15	10	7	4	0
Average obs.	14.44	4.38	2.19	0.88	0.47	0.31	0.22	0.12	0
per chart theor.	12.2	5.99	2.58	1.07	0.42	0.17	0.064	0.024	0.012

Table II.

Probable Frequencies of Groups. $\nu = 169$

m	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
2	1.96	0.022						
3	2.88	0.064	8×10^{-4}					
4	3.75	0.13	0.0029	3.5×10^{-5}				
5	4.58	0.21	0.0072	1.7×10^{-4}	2.1×10^{-6}			
6	5.37	0.30	0.014	5.0×10^{-4}	1.2×10^{-5}	1.5×10^{-7}		
7	6.12	0.41	0.023	0.0011	4.1×10^{-5}	1.0×10^{-6}	1.3×10^{-8}	
8	6.83	0.54	0.036	0.0021	1.0×10^{-4}	3.9×10^{-6}	9.7×10^{-8}	1.3×10^{-9}
9	7.49	0.68	0.053	0.0037	2.2×10^{-4}	1.1×10^{-5}	4.2×10^{-7}	1.1×10^{-8}
10	8.12	0.83	0.074	0.0061	4.2×10^{-4}	2.6×10^{-5}	1.3×10^{-6}	5.1×10^{-8}
11	8.71	0.99	0.10	0.0093	7.6×10^{-4}	5.5×10^{-5}	3.4×10^{-6}	1.7×10^{-7}
12	9.27	1.15	0.13	0.014	1.3×10^{-3}	1.1×10^{-4}	7.9×10^{-6}	4.9×10^{-7}
13	9.85	1.25	0.16	0.020	0.0021	2.0×10^{-4}	1.7×10^{-5}	1.2×10^{-6}
14	10.4	1.4	0.20	0.027	0.0031	3.4×10^{-4}	3.3×10^{-5}	2.8×10^{-6}
15	10.9	1.6	0.25	0.035	0.0046	5.4×10^{-4}	6.0×10^{-5}	5.8×10^{-6}
16	11.45	1.8	0.29	0.046	0.0064	8.4×10^{-4}	1.0×10^{-4}	1.1×10^{-5}
17	11.9	2.0	0.33	0.059	0.0088	0.0013	1.4×10^{-4}	2.0×10^{-5}
18	12.3	2.15	0.37	0.073	0.012	0.0019	2.0×10^{-4}	2.6×10^{-5}
19	12.65	2.35	0.43	0.089	0.016	0.0027	3.5×10^{-4}	4.4×10^{-5}
20	13.0	2.40	0.48	0.11	0.021	0.0037	5.6×10^{-4}	7.6×10^{-5}
21	13.3	2.6	0.53	0.13	0.027	0.0050	8.4×10^{-4}	1.3×10^{-4}
22	13.6	2.7	0.59	0.15	0.033	0.0066	0.0012	2.0×10^{-4}
23	13.9	2.8	0.66	0.18	0.040	0.0086	0.0017	3.1×10^{-4}
24	14.2	3.05	0.74	0.21	0.049	0.011	0.0023	4.6×10^{-4}
25	14.4	3.2	0.83	0.24	0.059	0.013	0.0031	6.5×10^{-4}
26	14.6	3.35	0·92	0.27	0.070	0.017	0.0040	9.1×10^{-4}
27	14.8	3.5	1.05	0.31	0.082	0.021	0.0052	0.0012
28	15.0	3.7	1.10	0.35	0.096	0.026	0.0066	0.0016
29	15.05	3.8	1.20	0.39	0.11	0.032	0.0084	0.0021
30	15.15	3.9	1.30	0.43	0.19	0.038	0.010	0.0028
31	15.3	4.0	1.40	0.48	0.22	0.045	0.013	0 0035
32	15.35	4.1	1.50	0.52	0.24	0.054	0.016	0.0046
33	15.5	4.2	1.60	0.56	0.26	0.064	0.020	0.0058
34	15.6	4.3	1.70	0.62	0.28	0.074	0.024	0.0073
35	15.7	4.3	1.80	0.67	0.31	0.085	0.029	0.0091
36	15.75	4.4	1.89	0.72	0.34	0.098	0.035	0.011
37	15.75	4.4	1.98	0.78	0.36	0.11	0.041	0.014
38	15.7	4.4	2.07	0.85	0.39	0.13	0.047	0.017
49	15.6	4.4	2.17	0.91	0.43	0.15	0.055	0.020
30	15.4	4.4	2.27	0.98	0.47	0.17	0.064	0.024

Table II. Continued.

m	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}
9	1.4×10^{-10}							
10	1.3×10^{-9}	1.7×10^{-11}						
11	6.8×10^{-9}	1.6×10^{-10}						
12	2.5×10^{-8}	8.5×10^{-10}						
13	7.7×10^{-8}	3.8×10^{-9}	1.1×10^{-10}					
14	2.1×10^{-7}	1.3×10^{-8}	5.3×10^{-10}					
15	5.0×10^{-7}	3.8×10^{-8}	2.2×10^{-9}					
16	1.1×10^{-6}	9.7×10^{-8}	7.3×10^{-9}	3.4×10^{-10}				
17	2.3×10^{-6}	2.3×10^{-7}	8.6×10^{-8}	1.1×10^{-9}				
18	4.3×10^{-6}	5.0×10^{-7}	1.2×10^{-7}	1.4×10^{-8}	1.9×10^{-10}			
19	6.7×10^{-6}	9.9×10^{-7}	1.7×10^{-7}	3.1×10^{-8}	2.4×10^{-9}			
20	1.1×10^{-5}	1.7×10^{-6}	2.8×10^{-7}	5.3×10^{-8}	7.3×10^{-9}	4.4×10^{-10}		
21	1.8×10^{-5}	3.0×10^{-6}	4.9×10^{-7}	8.9×10^{-8}	1.6×10^{-8}	1.4×10^{-9}		
22	3.2×10^{-5}	5.0×10^{-6}	8.5×10^{-7}	1.5×10^{-7}	2.8×10^{-8}	4.1×10^{-9}	2.8×10^{-10}	
23	5.2×10^{-5}	8.6×10^{-6}	1.4×10^{-6}	2.6×10^{-7}	4.9×10^{-8}	8.6×10^{-9}	8.3×10^{-10}	
24	8.3×10^{-5}	1.4×10^{-5}	2.5×10^{-6}	4.4×10^{-7}	8.6×10^{-8}	1.6×10^{-8}	1.7×10^{-9}	
25	1.3×10^{-4}	2.3×10^{-5}	4.2×10^{-6}	7.7×10^{-7}	1.5×10^{-7}	2.9×10^{-8}	4.6×10^{-9}	3.7×10^{-10}
26	1.9×10^{-4}	3.8×10^{-5}	7.0×10^{-6}	1.3×10^{-6}	2.6×10^{-7}	5.2×10^{-8}	9.6×10^{-9}	1.0×10^{-9}
27	2.7×10^{-4}	5.7×10^{-5}	1.2×10^{-5}	2.2×10^{-6}	4.4×10^{-7}	9.1×10^{-8}	1.9×10^{-8}	2.1×10^{-9}
28	3.8×10^{-4}	8.6×10^{-5}	1.9×10^{-5}	3.8×10^{-6}	7.5×10^{-7}	1.6×10^{-7}	3.4×10^{-8}	5.8×10^{-9}
29	5.3×10^{-4}	1.2×10^{-4}	2.9×10^{-5}	6.3×10^{-6}	1.3×10^{-6}	2.9×10^{-7}	7.0×10^{-8}	1.1×10^{-8}
30	7.2×10^{-4}	1.8×10^{-4}	4.3×10^{-5}	1.0×10^{-5}	2.2×10^{-6}	4.9×10^{-7}	1.1×10^{-7}	2.4×10^{-8}
31	9.8×10^{-4}	2.5×10^{-4}	6.4×10^{-5}	1.6×10^{-5}	3.6×10^{-6}	8.3×10^{-7}	1.9×10^{-7}	4.1×10^{-8}
32	0.0013	3.5×10^{-4}	9.1×10^{-5}	2.4×10^{-5}	5.9×10^{-6}	1.4×10^{-6}	3.3×10^{-7}	7.5×10^{-8}
33	0.0017	4.8×10^{-4}	1.3×10^{-4}	3.6×10^{-5}	9.4×10^{-6}	2.1×10^{-6}	5.5×10^{-7}	1.3×10^{-7}
34	0.0022	6.5×10^{-4}	1.8×10^{-4}	5.2×10^{-5}	1.4×10^{-5}	3.5×10^{-6}	8.8×10^{-7}	2.3×10^{-7}
35	0.0028	8.7×10^{-4}	2.5×10^{-4}	7.4×10^{-5}	2.1×10^{-5}	5.6×10^{-6}	1.4×10^{-6}	3.8×10^{-7}
36	0.0035	0.0012	3.5×10^{-4}	1.0×10^{-4}	3.1×10^{-5}	8.7×10^{-6}	2.3×10^{-6}	6.1×10^{-7}
37	0.0045	0.0015	4.9×10^{-4}	1.4×10^{-4}	4.4×10^{-5}	1.3×10^{-5}	3.6×10^{-6}	1.0×10^{-6}
38	0.0057	0.0019	6.5×10^{-4}	2.0×10^{-4}	6.2×10^{-5}	1.9×10^{-5}	5.6×10^{-6}	1.6×10^{-6}
39	0.0071	0.0024	8.4×10^{-4}	2.8×10^{-4}	8.8×10^{-5}	2.8×10^{-5}	8.5×10^{-6}	2.6×10^{-6}
40	0.0087	0.0031	0.0011	3.6×10^{-4}	1.2×10^{-4}	4.1×10^{-5}	1.3×10^{-5}	4.0×10^{-6}

Table II. Continued.

m	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}	x_{22}
28	5.0×10^{-10}					
29	1.4×10^{-9}					
30	3.6×10^{-9}	3.6×10^{-10}				
31	8.5×10^{-9}	9.2×10^{-10}				
32	1.6×10^{-8}	2.2×10^{-9}				
33	3.0×10^{-8}	5.8×10^{-9}	6.0×10^{-10}			
34	5.4×10^{-8}	1.2×10^{-8}	1.6×10^{-9}			
35	9.7×10^{-8}	2.3×10^{-8}	4.5×10^{-9}	4.7×10^{-10}		
36	1.7×10^{-7}	4.3×10^{-8}	7.0×10^{-9}	1.3×10^{-9}		
37	2.7×10^{-7}	7.7×10^{-8}	9.8×10^{-9}	3.8×10^{-9}	4.1×10^{-10}	
38	4.6×10^{-7}	1.3×10^{-7}	2.9×10^{-8}	5.5×10^{-9}	1.2×10^{-9}	
39	7.4×10^{-7}	2.3×10^{-7}	5.7×10^{-8}	1.2×10^{-8}	2.6×10^{-9}	4.0×10^{-10}
40	1.3×10^{-6}	3.6×10^{-7}	1.1×10^{-7}	2.5×10^{-8}	5.5×10^{-9}	8.7×10^{-10}

Table III.

Revised Data for the Irregularity of Stellar Distribution in the Paris Carte-du-Ciel Zone +24°.

Values of P_1, P_2 greater than 0.99999 are marked with an asterisk

Chart	Low density branch					High density branch				
	P_1	W_1	G_1	90% reality limit		P_2	W_2	G_2	90% reality limit	
				m	Groups				m	Groups
1	0	1.7	1.6	...	...	0.93	6.5	6.0	...	...
2	0	...	...	...	...	0	...	...	...	...
3	0.59	8.8	...	...	...	1.000	310.	2×10^4	34	13
4	0.67	16.	...	...	...	0.95	4.6	14.	...	...
5	0	...	...	...	...	0.64	1.5	2.1	...	...
6	0.89	44.	8.	...	...	0 96	22.	5.2	...	...
7	0	1.6	1.5	...	...	0	1.0	1.1	...	...
8	0.93	10^4	2.6	...	...	0.92	12.	3.3	...	...
9	0.975	8.8	190.	32	9	0.98	24.	66.	30	8
10	0	...	...	...	...	0	1.5	...	...	...
11	0.74	26.	2.2	...	...	0.92	11.	2.0	...	...
12	0	...	...	...	...	0.79	1.1	5.0	...	...
13	0.75	1.7	28.	...	...	0.79	...	5.5	...	...
14	0.62	6.7	2.5	...	...	0.78	8.0	1.8	...	...
15	0	...	...	...	...	0	1.8	...	...	...
16	0.50	4.4	1.0	...	...	0.90	3.5	7.3	...	...
17	0.83	2.4	30.	...	...	0.93	38.	4.0	...	...
18	0	1.6	1.0	...	...	0	1.0	...	...	...
19	0.96	2.9	130.	24	7	0.999	8.0	770.	30	9 ; 6
20	0	...	...	...	...	0.64	1.4	3.5	...	...
21	0.69	10.	3.8	...	...	0.992	22.	150.	36	10
22	0.999	170.	900.	20	7	0.999	11.	6×10^4	36	15
23	0.96	1.2	380.	7	4	0.97	22.	12.	...	...
24	0.60	1.2	12.	...	...	0.82	3.2	3.0	...	...
25	0.46	2.0	3.5	...	...	0.97	6.0	40.	14	4 ; 3
26	1.000*	6.0	10^6	34	17	0.996	...	3000.	21	8
27	1.000	720.	9000.	29	11	0.990	50.	76.	4	3
28	0.71	4.8	5.0	...	...	0.990	460.	8.	...	...
29	1.000*	46.	10^6	30	8 ; 7 ; 6	0.94	21.	3.8	...	...
30	1.000	12.	1500.	25	9	0.83	1.2	4.0	...	...
31	1.000*	10^{30}	10^9	28	18	1.000*	10^6	10^{14}	36	27
32	1.000*	10^6	4.9	...	...	0.994	86.	4.7	...	...
33	1.000*	10^{17}	10^9	39	22	1.000*	10^{10}	4×10^4	31	13
34	1.000*	10.	10^9	40	23	0.98	4.7	5.5	...	...
35	1.000*	10^5	10^5	34	14	1.000*	750.	10^8	38	21
36	1.000*	10^{112}	10^{20}	39	35	1.000*	10^{42}	1600.	9	5
37	1.000*	3000.	10^6	24	13	1.000	500.	100.	37	10
38	1.000*	10^8	5000.	33	12	1.000*	10^{22}	10^{14}	35	24
39	0.988	20.	27.	...	...	0.998	33.	33.	11	4
40	1.000*	10^{14}	10^8	33	18	1.000*	10^5	4×10^4	38	14

Table III. Continued.

Chart	Low density branch					High density branch				
	P_1	W_1	G_1	90 % reality limit		P_2	W_2	G_2	90 % reality limit	
				m	Groups				m	Groups
41	0.61	3.3	3.3	...	...	0.96	3.1	2.5	...	...
42	1.000*	10^{20}	10^{12}	30	21	1.000*	10^{21}	10^{10}	37	22
43	0.98	64.	4.0	...	...	1.000*	10^{6}	10^{9}	36	21
44	1.000*	10^{7}	10^{5}	37	8; 8	1.000*	10^{6}	17.	20	5; 4
45	1.000	35.	1000.	30	10	1.000*	10^{8}	8.5	21	3; 3; 3
46 I	0.999	52.	360.	26	8	1.000	12.	10^{4}	27	8; 6; 5
II	1.000*	10^{7}	10^{5}	34	15	1.000*	10^{7}	10^{9}	29	18
III	1.000*	2500.	3000.	25	9	1.000*	10^{5}	350.	11	5
IV	1.000*	10^{18}	10^{7}	21	13	1.000*	10^{37}	10^{13}	22	19
47 I	1.000*	10^{11}	10^{10}	32	20	1.000*	80.	10^{5}	34	14
II	0	1.6	...	...	...	0.90	10.	1.5	...	...
III	1.000*	10^{5}	10^{6}	40	19	1.000*	10^{13}	10^{12}	37	25
IV	0.91	...	40.	11	4	1.000	3.5	360.	12	5
48 I	1.000	1.1	3600.	36	12	0.999	500.	8.0	18	4
II	1.000*	720.	2×10^{4}	17	8	1.000	1.6	900.	36	9; 7
III	0.994	8.8	97.	39	8; 6	0.83	3.2	...	...	...
IV	0	1.3	...	...	...	0.87	5.9	1.5	...	...
49 I	1.000*	3000.	10^{5}	35	15	0.96	3.1	3.4	...	...
II	0.94	90.	1.9	...	...	1.000*	33.	10^{6}	24	6; 5; 5
III	0.999	8.3	340.	34	10	1.000	3.2	1000.	9	5
IV	0.70	4.3	1.5	...	...	0.997	108.	10.	21	5; 4
50 I	0.99	26.	45.	32	8	1.000	4.3	500.	36	11
II	1.000*	1.8	10^{9}	29	10; 9	0.96	2.6	3.6	...	...
III	1.000*	3000.	3×10^{4}	19	9	1.000*	10^{6}	10^{5}	35	15
IV	0.998	80.	200.	8	4	0.992	7.0	26.	6	3
51 I	1.000*	2×10^{4}	10^{8}	40	21	0.997	24.	53.	4	3
II	1.000*	2000.	10^{7}	21	13	0.997	53.	20.	29	7
III	1.000*	1000.	10^{6}	33	16	0.999	120.	60.	20	3; 3; 3; 3
IV	1.000	58.	1000.	22	8	0.94	1.7	27.	12	3; 3
52 I	0.984	10.	21.	...	...	1.000	3.2	400.	29	9
II	0.91	5.1	7.8	...	...	1.000	1.0	3000.	27	10
III	0.97	1.5	190.	28	8	0.94	1.7	26.	30	7
IV	1.000	7.4	2×10^{4}	33	13	0.999	4000.	1.4	...	...
53 I	0.96	12.	5.7	...	...	1.000	49.	380.	24	8
II	1.000	4.1	850.	28	9	0.998	3.1	140.	36	6; 6; 5; 5
III	0.93	39.	...	...	...	1.000	4.2	700.	23	6; 5
IV	1.000	5.0	770.	32	10	0.997	9.8	75.	34	9
54 I	0.98	24.	15.	...	...	0.83	3.2	1.5	...	...
II	0.61	2.1	3.0	...	...	0.96	1.2	41.	31	6; 5; 5
III	0.93	8.8	3.1	...	...	0.97	1.9	70.	35	9
IV	1.000	43.	2×10^{4}	35	14	1.000*	6000.	1200.	33	11
55	1.000	800.	70.	35	9	1.000	135.	200.	31	8; 6
56	1.000*	530.	10^{5}	37	9; 8; 7;	1.000*	10^{6}	3.4	...	...
57	0.83	1.5	9.5	...	...	0.98	15.	4.2	...	...

Table III Continued.

Chart	Low density branch					High density branch				
	P_1	W_1	G_1	90% reality limit		P_2	W_2	G_2	90% reality limit	
				m	Groups				m	Groups
58	1.000*	10^5	10^4	38	14	1.000*	10^6	10^7	32	10; 7
59	0.31	2.9	1.0	...	...	0.64	1.2	2.5	...	...
60	0.90	12.	18.	...	...	0.96	8.1	8.8	...	...
61	0.83	3.6	19.	...	...	0.998	2.2	1000.	13	6
62	0.23	1.9	2.5	...	...	0.49	3.5	1.0	...	...
63	0.998	26.	2000.	25	9	0.998	3.1	1000.	30	10
64	0	1.5	...	...	...	0.64	1.7	2.0	...	...
65	0.95	12.	68.	...	...	0.95	6.0	11.	...	...
66	0.50	4.0	...	...	...	0.95	8.0	5.0	...	...
67	0.75	24.	3.3	...	...	0.79	1.4	4.6	...	...
68	0.60	2.8	5.6	...	...	0.69	5.2	...	...	...
69	0.88	4.4	30.	...	...	0.996	...	1000.	26	8; 6
70	0	...	...	...	...	0.69	6.7	1.9	...	...
71	0	...	...	...	...	0	1.1	...	...	...
72	0	9.5	...	...	...	0	5.2	1.7	...	...
73	0.94	1.9	700.	17	6	0.20	1.6	2.4	...	...
74	0	1.7	...	...	...	0.20	1.8	2.0	...	...
75	0	1.2	...	...	...	0.60	1.2	10.	...	...
76	0	...	...	...	...	0.20	1.2	2.5	...	...
77	0	1.1	3.5	...	...	0	...	...	...	...
78	0	...	...	...	...	0	...	...	...	...
79	0	...	...	...	...	0.20	1.5	3.5	...	...
80	0	...	...	...	...	0	...	...	...	...
81	0	3.3	...	...	...	0	2.2	...	...	...
82	0	...	...	...	...	0	...	...	...	...
83	0	...	...	...	...	0	...	...	...	...
84	0	...	...	...	...	0	...	...	...	...
85	0	...	...	...	...	0	2.7	...	...	...
86	0	4.0	5.6	...	...	0	3.4	1.1	...	...
87	0	...	...	...	...	0.76	15.	17.	...	...
88	0	2.1	...	...	...	0.76	2.8	18.	...	...
89	0	1.5	1.7	...	...	0.40	9.0	5.	...	...
90	0	...	...	...	...	0.20	...	3.5	...	...
91	0	3.5	...	...	...	0.80	260.	...	...	...
92	0	...	...	...	...	0.60	1.1	12.	...	...
93	0	12.	3.5	...	...	0.98	7.6	3000.	20	4; 4; 4
94	0	2.4	1.6	...	...	0.80	...	55.	...	...
95	0	1.7	...	...	...	0.20	1.6	3.1	...	...
96	0	1.5	...	...	...	0	...	...	...	...
97	0	4.0	...	...	...	0	1.8	...	...	...
98	0	...	...	...	...	0	...	...	...	...
99	0	4.0	...	...	...	0	1.2	...	...	...
100	0	...	...	...	...	0	1.2	...	...	...

Table III. Continued.

Chart	Low density branch					High density branch				
	P_1	W_1	G_1	90% reality limit		P_2	W_2	G_2	90% reality limit	
				m	Groups				m	Groups
101	0	4.0	...	...	...	0.20	3.4	3.0	...	...
102	0	...	...	...	...	0	...	...	...	...
103	0	5.0	...	...	...	0.20	10.	3.3	...	...
104	0	2.1	1.6	...	...	0.40	1.5	4.0	...	...
105	0	...	...	...	...	0.76	1.2	21.	...	...
106	0	...	...	...	...	0	2.4	...	...	...
107	0	1.9	1.2	...	...	0	2.1	...	...	...
108	0	6.0	...	...	...	0	1.3	...	...	...
109	0	3.5	...	...	...	0	...	...	...	...
110	0	4.0	4.0	...	...	0.80	20.	60.	...	...
111	0	...	...	...	...	0	...	...	...	...
112	0	1.1	4.0	...	...	0.86	3.1	140.	...	...
113	0	...	...	...	...	0	...	...	...	...
114	0	...	...	...	...	0.90	8.1	350.	27	8
115	0	1.0	...	...	...	0.60	2.2	8.0	...	...
116	0	...	...	...	...	0.20	1.2	3.3	...	...
117	0	2.1	3.5	...	...	0.20	1.6	2.4	...	...
118	0	2.5	...	...	...	0.40	12.5	5.0	...	...
119	0	...	...	...	...	0.98	1.1	1200.	26	9
120	0	1.6	...	...	...	0	...	...	...	...
121	0	4.0	...	...	...	0.20	1.0	2.2	...	...
122	0.59	10.	...	...	...	0.69	4.1	...	...	...
123	0.50	6.7	...	...	...	0.96	5.3	18.	...	...
124	0.47	3.2	3.3	...	...	0.95	12.	4.4	...	...
125	0.47	2.8	3.0	...	...	0.97	5.0	60.	13	3; 3
126	0.88	3.2	48.	...	...	0.92	3.8	14.	...	...
127	1.000*	100.	10^5	34	8; 8	0.986	21.	70.	39	8; 7; 6
128	0.92	3.0	100.	...	...	0.49	2.2	...	...	...
129	0.23	1.7	3.5	...	...	0.98	15.	55.	21	6
130	0.98	4.2	280.	23	7	0.88	1.4	26.	...	...
131	1.000*	9.5	10^6	22	12	0.93	6.7	6.8	...	...
132	0.23	1.7	3.3	...	...	0.69	5.9	...	...	...
133	0.92	17.	27.	...	...	0.99	160.	41.	21	6
134	0	...	...	...	...	0.64	...	3.0	...	...
135	0.985	210.	4.2	...	...	1.000*	110.	5000.	28	8; 6
136	0.97	5.3	42.	29	7	1.000	15.	1000.	18	7
137	1.000*	93.	10^9	35	21	1.000	15.	5×10^4	11	7
138	0.992	3.5	350.	37	11	1.000*	10^6	200.	8	4
139	1.000*	97.	10^5	40	15; 9	0.93	21.	1.7	...	...
140	1.000*	10^5	10.	...	...	1.000*	10^4	4000.	30	11
141 I	0.95	1.4	97.	34	9	0.97	2.8	6.2	...	...
II	1.000	14.	700.	28	9	0.98	16.	2.0	...	...

Table III. Continued.

Chart	Low density branch					High density branch				
	P_1	W_1	G_1	90% reality limit		P_2	W_2	G_2	90% reality limit	
				m	Groups				m	Groups
141 III	0.95	21.	3.3	...	...	1.000*	4.4	10^{5}	13	6; 4
IV	1.000*	7.4	10^{5}	31	8; 7	0.999	6.1	250.	35	10
142	1.000*	10^{13}	10^{7}	35	12; 9	1.000*	10^{14}	10^{7}	29	15
143 I	1.000*	1.8	10^{5}	30	9; 7	0.998	3.5	200.	38	7; 6; 6
II	1.000*	1.6	10^{7}	40	20	1.000	5.5	600.	14	6
III	0.70	1.2	4.4	...	...	1.000	5.5	260.	26	8
IV	0.25	2.0	...	...	...	1.000	17.	2500.	31	11
144	1.000*	10^{17}	4000.	15	4; 4	1.000*	10^{11}	10^{13}	28	21
145	1.000*	1500.	2000.	33	11	1.000*	10^{6}	700.	31	8
146	1.000*	10^{78}	10^{9}	37	18; 9	1.000*	10^{36}	10^{20}	37	32
147	1.000*	10^{34}	10^{9}	39	23	1.000*	10^{36}	10^{22}	39	18; 17
148	1.000*	10^{44}	2×10^{4}	34	13	1.000*	10^{37}	10^{9}	37	21
149	1.000*	10^{6}	10^{6}	16	5; 4; 4	1.000*	10^{9}	10^{5}	18	5; 5
150	1.000*	10^{5}	10^{5}	32	14	1.000	2×10^{4}	120.	32	6; 6
151	1.000	2000.	44.	23	6	1.000*	10^{5}	10^{8}	20	13
152	0.84	5.2	2.8	...	...	0.98	22.	2.7	...	...
153	1.000*	10^{26}	10^{7}	32	17	1.000*	10^{14}	3×10^{4}	33	13
154	1.000*	10^{7}	10^{10}	33	21	1.000*	10^{5}	10^{11}	36	14; 11
155	1.000*	10^{8}	4000.	38	13	1.000*	10^{8}	21.	36	6; 6
156 I	0.91	7.1	4.7	...	...	1.000	1.1	2×10^{4}	39	15
II	0	1.1	1.9	...	...	0.98	2.2	8.0	4	2
III	0.994	11.	83.	34	7; 6	1.000*	2.3	10^{10}	37	22
IV	1.000	720.	9.6	...	...	1.000*	170.	10^{6}	28	14
157	1.000*	1500.	10^{10}	39	23	1.000*	10^{6}	1500.	12	6
158	0.999	1000.	1.7	...	...	0.98	9.8	2.4	..	...
159	1.000*	16.	10^{5}	24	7; 6	1.000*	250.	10^{6}	34	16
160	1.000	3000.	12.	...	...	0.998	145.	8.3	32	7
161	0.87	8.6	...	...	...	0.97	2.1	5.5	...	...
162	1.000	5.3	2×10^{4}	30	12	0.987	5.8	8.0	21	5
163	1.000	25.	1000.	34	11	1.000*	2000.	5000.	15	4; 4
164	0.93	10^{4}	2.2	...	...	0.986	4000.	2.5	...	...
165	0.67	17.	1.2	...	...	0.997	6000.	80.	31	8
166	0	...	...	...	...	0.999	7.0	2×10^{4}	27	11
167	1.000*	2×10^{4}	10^{5}	35	9; 8	0.991	75.	90.	27	5; 5
168	0.31	2.8	1.9	...	...	0.992	30.	200.	39	11
169	0	1.7	...	...	...	0	...	...	...	...
170	0.997	4.0	5×10^{4}	31	13	0.84	1.3	14.	...	...
171	1.000	700.	1600.	39	13	0.997	17.	300.	25	8
172	0.96	160.	17.	...	...	0.98	170.	6.1	...	...

Table III. Continued.

Chart	Low density branch					High density branch				
	P_1	W_1	G_1	90% reality limit		P_2	W_2	G_2	90% reality limit	
				m	Groups				m	Groups
173	0.60	1.3	11.	...	...	0.99	2.1	330.	13	5
174	0.31	2.6	...	...	...	0.998	3.3	1000.	18	8
175	0.60	1.7	8.0	...	...	0.84	...	8.0	...	...
176	0.98	26.	200.	31	9	0.984	5.4	130.	38	8; 7
177	0.31	2.2	...	...	...	0.49	3.4	1.4	...	...
178	0.47	2.6	3.2	...	...	0.82	2.0	3.3	...	...
179	0.83	3.1	17.	...	...	0.996	1.6	600.	32	10
180	0.23	...	2.1	...	...	0.998	3.1	4000.	34	12

With the aid of table II, weights of geometrical configurations were computed for all charts showing a positive excess in the Paris zone. The definition of the "group weights" G_1 und G_2 is identical, and their calculation very similar to the calculation of the weights W_1, W_2 of positive excesses in observed frequency tables. The positive excess in geometrical configurations reveals itself in the appearance of groups of higher order in a number greater than expected from table II, accompanied by a deficiency of x_1. Geometrical configurations were considered separately for all density classes which showed a positive excess of frequency, and for which $m \leqslant 40$. In this way, one and the same branch, e. g. the low density branch, may yield several values of m which are to be treated independently. The calculation of the effective number of independent experiments is somewhat more complicated than in the case of the distribution of densities: generally formula (1) is valid for a given density group, except that the ratios $\frac{n_c}{n'_c}$ are to be multiplied by $\frac{n'_c - n''_c}{n'_c}$, when $n''_c > \frac{1}{2}$; n'_c denotes the cumulative sum for a given group of i squares and given m; n''_c — the cumulative sum for the group of the same denomination i, and for $m_1 < m$, or for the next preceding density group considered.

As an example, we take the low density branch of Paris Chart 168 considered before. Evidently, by force of the restrictions mentioned above, we have to consider geometrical configurations in two cases: $m_1 = 15$, $r \leqslant 3$; and $m_2 = 31$, $r \leqslant 4$. The results are as follows:

	a) $m_1 = 15$; $r \leqslant 3$.			b) $m_2 = 31$; $r \leqslant 4$.			
Group	$\geqslant x_3$	x_2	x_1	$\geqslant x_4$	x_3	x_2	x_1
Frequency obs. n_0	1	2	8	1	3	3	12
Frequency comp. n_c	0.29	1.6	10.9	0.76	1.4	4.0	15.3
$n'_0 = \Sigma n_0$	1	3	...	1	4	7	...
$n'_c = \Sigma n_c$	0.29	1.9	...	0.76	2.2	6.2	...
Excess, $n'_0 - n'_c$	+0.71	+1.1	...	+0.24	+1.8	+0.8	...
c	1.5	1.6	...	1.9	2.6	4.0	...
$x = \frac{n'_0 - n'_c}{c}$	0.5	0.4	...	0.1	0.7	0.2	...
Group-weight, G	4	3.5	...	2	6	2.5	...

Here G is the group weight, corresponding to W of table 8 of the first paper. We have $G_{max} = 6$. The effective number of independent experiments is 1 (from x_4, m_2) $+ \frac{1.4}{2.2} (x_3, m_2) +$ $+ \frac{4.0}{6 \cdot 2} \times \frac{(6.2 - 1.9)}{6.2} (x_2, m_2) + 1$ (from x_2, m_1) $= 3.2$. The final weight of the geometrical configurations in the low density branch is thus $G_1 = \frac{6}{3.2} = 1.9$, in agreement with formula (2).

IV. The Irregularity of Stellar Distribution in the Paris Zone $+24^0$.

Table III contains a summary of the revised data for the Paris Zone. Supplementary data to this table are contained in Table 11 of the first paper, and in the original counts by Miss Lukk. The first column of the table gives the ordinal number of the Paris chart. The next five columns refer to the ascending branch, or the low density portion of the distribution of densities. The second column gives P_1, the "probability of reality" of irregularities on the low density branch; computed in a manner explained below, this quantity represents a combined probability derived from both criteria, the distribution of densities and geometrical configurations. Next follow W_1, the weight of the positive excess, and G_1, the weight of the geometrical configuration on the low density branch; weights smaller than 1 are not printed; also, G_1 was mostly not calculated when $W_1 < 1$. The fifth and sixth columns refer to cases when vacancies, or adjacent groups of low density found on the charts showed a geometrical probability of reality equal to, or greater than 0.90; this probability of reality, depending directly repon G_1, and denoted by q_1, is computed according to a method explained below; m denotes the number of selected squares of low density; the figures in the sixth column indicate the character of the unusual adjacent groups found on the charts; e. g. chart 171 shows a vacancy of thirteen adjacent squares of low density, chart 149 has one group of five, and two groups of four squares, etc. The majority of these unusual groups may be considered as individually real, representing either real vacancies, or obscuring clouds (maxima of absorption). The last five columns refer to analogous data for the descending branch, or the high density portion of the distribution: P_2, the probability of reality of irregularities on the high density branch, etc. Several rich MilkyWay maps were divided into quadrants in the first paper; the data for the quadrants are given in Table III separately.

Before discussing the data, we remind that the absolute values of the weights W_1 and W_2, for the same kind of irregularity, depend considerably upon the total number of stars per chart, the weights increasing with increasing number of stars; because of the high spurious weights in the first paper, this circumstance offered a serious obstacle to the direct intercomparison of regions differing in galactic latitude. In table III, the spurious weights must be small or negligible, as has been shown above, and a direct intercomparison of different galactic zones seems therefore permissible; that the effect of spurious weights has no influence in the present case, follows from the fact that the behaviour of the W in the different galactic zones is very well checked by the behaviour of the G (compare Tables IV to X below); now, these latter do not depend upon the number of stars, but only upon the number of squares which is constant all over the zone.

Tables IV to IX contain a summary of the distribution of the two characteristics of irregularity, W and G, in the Paris zone, separately for different limits of galactic latitude. The theoretical distribution of one kind of weight in the case of pure chance is given by formula (4) as follows:

Table A.

Theoretical Frequency of Weights

Limits of W or G	<1	1—2	2—4	4—8	8—16	16—32	32—64	64—128	> 128	Sum
Relative frequency, f	0.368	0.239	0.173	0.102	0.057	0.030	0.015	0.008	0.008	1.000

Table IV.

Distribution of W_1, G_1 in the Paris Zone, Galactic Latitude $+45^0$ to $+87^0$. Charts Nr. 72 to 121 incl.

G_1 ↓	W_1 →													
	<1	1—2	2—4	4—8	8—16	16—32	32—64	64—128	128—256	256—512	>512	All, n_0	fS (S = 29)	q_1
Not. comp.	21	0	0	0	0	0	0	0	0	0	0	21	...	...
<1	0	6	5	6	1	0	0	0	0	0	0	18	10.6	0
1—2	0	2	2	0	0	0	0	0	0	0	0	4	6.9	0
2—4	0	1	1	0	1	0	0	0	0	0	0	3	5.0	0
4—8	0	1	0	2	0	0	0	0	0	0	0	3	3.0	0
8—16	0	0	0	0	0	0	0	0	0	0	0	0	1.6	...
16—32	0	0	0	0	0	0	0	0	0	0	0	0	0.9	...
32—64	0	0	0	0	0	0	0	0	0	0	0	0	0.4	...
64—128	0	0	0	0	0	0	0	0	0	0	0	0	0.23	...
128—256	0	0	0	0	0	0	0	0	0	0	0	0	0.12	...
256—512	0	0	0	0	0	0	0	0	0	0	0	0	0.06	...
>512	0	1	0	0	0	0	0	0	0	0	0	1	0.06	0.94
All, n_0	21	11	8	8	2	0	0	0	0	0	0	50	...	...
fS (S = 50)	18.4	12.0	8.6	5.1	2.8	1.5	0.8	0.4	0.2	0.1	0.1	...	...	...
p_1	0	0	0	0	0	...	...	...	...	...	...	...	...	...

Table V.

Distribution of W_2, G_2 in the Paris Zone, Galactic Latitude $+45^0$ to $+87^0$. Charts Nr. 72 to 121 incl.

G_2 ↓	W_2 →													
	<1	1—2	2—4	4—8	8—16	16—32	32—64	64—128	128—256	256—512	>512	All, n_0	fS (S = 23)	q_2
Not comp.	13	0	0	0	0	0	0	0	0	0	0	13	...	...
<1	1	3	4	0	0	0	0	0	0	1	0	9	8.5	0
1—2	0	0	1	1	0	0	0	0	0	0	0	2	5.5	0
2—4	1	8	1	0	1	0	0	0	0	0	0	11	4.0	0.20
4—8	0	1	0	0	2	0	0	0	0	0	0	3	2.3	0.40
8—16	0	2	1	0	0	0	0	0	0	0	0	3	1.3	0.60
16—32	0	1	1	0	1	0	0	0	0	0	0	3	0.7	0.76
32—64	1	0	0	0	0	1	0	0	0	0	0	2	0.35	0.80
64—128	0	0	0	0	0	0	0	0	0	0	0	0	0.18	0.83
128—256	0	0	1	0	0	0	0	0	0	0	0	1	0.09	0.86
256—512	0	0	0	0	1	0	0	0	0	0	0	1	0.04	0.90
>512	0	1	0	1	0	0	0	0	0	0	0	2	0.04	0.98
All, n_0	16	16	9	2	5	1	0	0	0	1	0	50	...	...
fS (S = 50)	18.4	12.0	8.6	5.1	2.8	1.5	0.8	0.4	0.2	0.1	0.1	...	...	...
p_2	0	0	0	0	0	0	...	...	...	0.80	...	...	...	...

Table VI.

Distribution of W_1, G_1 in the Paris Zone, Galactic Latitude $+23^0$ to $+44^0$. Charts Nr. 1 to 29; 59 to 71; 122 to 134; 164 to 180 incl.

G_1 ↓	W_1 →													
	< 1	1—2	2—4	4—8	8—16	16—32	32—64	64—128	128—256	256—512	> 512	All, n_0	fS (S = 31)	q_1
Not. comp.	10	0	0	0	0	0	0	0	0	0	0	10	...	...
< 1	0	2	2	2	3	0	0	0	0	0	0	9	11.4	0
1—2	0	3	2	1	0	1	0	0	0	0	0	7	7.4	0
2—4	1	3	4	1	1	2	0	0	0	0	2	14	5.4	0.23
4—8	0	0	1	1	0	0	0	0	0	0	0	2	3.2	0.42
8—16	0	3	0	0	0	0	1	0	0	0	0	4	1.8	0.60
16—32	0	1	3	1	1	1	0	0	1	0	0	8	0.9	0.75
32—64	0	0	1	0	0	0	0	0	0	0	0	1	0.46	0.83
64—128	0	0	1	0	1	0	0	0	0	0	0	2	0.25	0.89
128—256	0	0	1	0	1	1	0	0	0	0	0	3	0.12	0.94
256—512	0	1	0	1	0	0	0	0	0	0	0	2	0.06	0.96
> 512	0	0	0	2	1	1	1	1	1	0	3	10	0.06	0.994
All, n_0	11	13	15	9	8	6	2	1	2	0	5	72	...	...
fS (S = 44)	16.2	10.5	7.6	4.5	2.5	1.3	0.66	0.35	0.18	0.09	0.09	...	...	...
p_1	0	0	0.31	0.50	0.59	0.67	0.73	0.80	0.83	0.87	0.91	...	...	...

Table VII.

Distribution of W_2, G_2 in the Paris Zone, Galactic Latitude $+23^0$ to $+44^0$. Charts Nr. 1 to 29; 59 to 71; 122 to 134; 164 to 180 incl.

G_2 ↓	W_2 →													
	< 1	1—2	2—4	4—8	8—16	16—32	32—64	64—128	128—256	256—512	> 512	All, n_0	fS (S = 23)	q_2
Not. comp.	1	0	0	0	0	0	0	0	0	0	0	1	...	...
< 1	1	4	1	3	0	0	0	0	0	0	0	9	8.5	0
1—2	0	1	2	1	1	0	0	0	0	0	0	5	5.5	0
2—4	1	4	2	0	2	1	0	0	0	0	1	11	4.0	0.64
4—8	1	2	1	2	2	1	1	0	1	0	0	11	2.3	0.79
8—16	1	1	1	2	1	1	0	0	0	1	0	8	1.3	0.84
16—32	0	1	0	1	0	0	0	0	0	0	0	2	0.7	0.88
32—64	0	0	0	2	1	0	0	0	1	0	0	4	0.35	0.90
64—128	0	0	0	0	0	2	1	1	0	0	1	5	0.18	0.92
128—256	0	0	0	1	0	2	0	0	0	0	0	3	0.09	0.95
256—512	0	0	1	0	0	1	0	0	0	0	0	2	0.04	0.98
> 512	2	1	4	1	2	0	0	0	0	1	0	11	0.04	0.996
All, n_0	7	14	12	13	9	8	2	1	2	2	2	72	...	...
fS (S = 35)	12.9	8.4	6.1	3.6	2.0	1.0	0.5	0.28	0.14	0.07	0.07	...	...	...
p_2	0	0	0.49	0.69	0.78	0.83	0.86	0.89	0.91	0.94	0.96	...	...	...

Table VIII.

Distribution of W_1, G_1 in the Paris Zone, Galactic Latitude $+0^0$ to $+22^0$. Charts Nr. 30 to 58, and 135 to 163 incl.

Quadrants of charts 46 to 54, 141, 143 and 156 are counted separately

G_1 ↓	W_1 →												fS (S = 15)	q_1
	<1	1—2	2—4	4—8	8—16	16—32	32—64	64—128	128—256	256—512	>512	All, n_0		
<1	0	2	1	0	1	0	1	0	0	0	0	5	5.5	0
1—2	0	1	0	1	0	0	0	1	0	0	1	4	3.6	0
2—4	0	0	2	1	1	1	0	0	0	0	0	5	2.6	0.48
4—8	0	1	0	2	1	0	1	0	1	0	1	7	1.5	0.70
8—16	0	1	0	0	0	1	0	0	0	0	3	5	0.85	0.83
16—32	0	0	0	0	1	1	0	0	0	0	0	2	0.45	0.88
32—64	1	0	0	1	0	1	0	0	0	0	1	4	0.22	0.91
64—128	0	1	0	0	2	0	0	0	0	0	1	4	0.12	0.95
128—256	0	1	0	0	0	0	0	1	0	0	0	2	0.06	0.97
256—512	0	0	1	0	1	0	1	0	0	0	0	3	0.03	0.99
>512	0	4	0	5	3	2	3	2	0	0	34	53	0.03	0.9994
All, n_0	1	11	4	10	10	6	6	4	1	0	41	94	...	...
fS (S = 20)	7.4	4.8	3.5	2.0	1.1	0.6	0.3	0.16	0.08	0.04	0 04	...	...	...
p_1	0	0	0.25	0.70	0.87	0.90	0.93	0.94	0.95	0.96	0.999	...	...	...

Table IX.

Distribution of W_2, G_2 in the Paris Zone, Galactic Latitude $+0^0$ to $+22^0$. Charts Nr. 30 to 58, and 135 to 163 incl.

Quadrants of charts 46 to 54, 141, 143 and 156 are counted separately

G_2 ↓	W_2 →												fS (S = 10)	q_2
	<1	1—2	2—4	4—8	8—16	16—32	32—64	64—128	128—256	256—512	>512	All, n_0		
<1	0	0	1	0	0	0	0	0	0	0	0	1	3.7	0
1—2	0	0	1	1	1	1	0	0	0	0	1	5	2.4	0
2—4	0	0	3	0	2	1	0	0	0	0	1	7	1.7	0.76
4—8	0	1	2	1	1	0	0	1	0	0	0	6	1.0	0.83
8—16	0	0	1	1	0	0	0	1	1	1	1	6	0.6	0.90
16—32	0	2	0	1	0	0	1	0	0	0	1	5	0.3	0.94
32—64	0	1	0	0	0	1	1	1	0	0	0	4	0.15	0.96
64—128	0	1	0	0	1	0	0	0	0	0	1	3	0.08	0.97
128—256	0	0	2	1	0	0	0	0	1	0	1	5	0.04	0.99
256 512	0	0	2	2	0	0	1	0	0	0	1	6	0.02	0.997
>512	0	3	2	3	3	2	1	3	2	0	27	46	0.02	0.9996
All, n_0	0	8	14	10	8	5	4	6	4	1	34	94	...	...
fS (S = 13)	4.8	3.1	2.3	1 3.	0.74	0.39	0.20	0.10	0.05	0.025	0.025	...	...	...
p_2	0	0	0.83	0.87	0.90	0.93	0.95	0.97	0.98	0.99	0.9993	...	...	...

Real irregularities in stellar distribution should reveal themselves through an excess in the frequency of large weights, and a deficiency of small weights. From a comparison of the observed and the theoretical distribution, it is possible to draw conclusions referring to the frequency of real irregularities, and to the probability of existence of real irregularities in an individual case; this probability we will call further "the probability of reality". Let p_1, p_2 denote the probabilities of reality depending upon W_1, W_2, and q_1, q_2 — those depending upon G_1, G_2. Let us further consider one kind of weights, e. g. W_1; let T be the total number of charts, and let a number S among them be void of real irregularities able to affect the weights W_1. For a given value of W_1 (or, rather, for given more or less narrow limits of W_1) let the observed number be n_0; the probability of reality is then given by

$$p_1 = \frac{n_0 - fS}{n_0} \quad \ldots\ldots \quad (6),$$

where f is given by Table A; for S we must find some approximate value.

It is safe to underestimate the probability of reality; therefore, a maximum value of S in (6) will answer our purposes the best. Such a maximum value we obtain by assuming that all weights below 2 are purely accidental; Let the number of weights below 2 be n_2; according to Table A, n_2 should be equal to $(0.368 + 0.239)$ S, whence

$$S_{max} = 1.65\ n_2 \ \ldots \ (7),$$

observing that $S \leqslant T$.

This is a maximum value, because some of the weights less than 2 may correspond to charts with real irregularities.

In the distribution tables IV to IX, the data connected with formulae (7) and (6) are given. The probabilities p, q, are not directly computed values, but are smoothed on the assumption of a steady increase of p or q with the increase of the corresponding weight. For weights below 2, p and q are put equal to zero as a consequence of our assumption connected with formula (7). Although underestimated, the probabilities p, q may be regarded as close to actual values, except perhaps for galactic latitudes below 23^0 where it appears probable that practically all charts show real irregularities. Nevertheless, also in this case we thought it safer to assume minimum values of the probabilities; this procedure evidently means disregarding some less pronounced irregularities, in other words, it is a question of selection.

In table IV we may notice, that the distribution of W_1 is very close to the theoretical, and G_1 shows rather a "negative excess", except for one outstanding value; the agreement with theory, and still more so the tendency toward an excess in the frequency of low weights, may serve as indication of the insignificance of spurious, "observational" weights in our present method of computation. At the same time, we conclude that real irregularities in the low density branch are very rare or absent in these high galactic latitudes; this conclusion is equivalent to a negative evidence for the existence of any important obscuring clouds in high galactic latitudes.

Table V shows, that in the same high galactic latitude region irregularities affecting the high density branch are not negligible; considering the negative evidence of table IV, we know that these irregularities cannot be a secondary effect of obscuration; it stands thus without doubt that real irregularity in the distribution of stars in space, — "clustering" —, plays a conspicuous rôle in the apparent arrangement of stars in high galactic latitudes.

Taking into account that the weights W_1, G_1 are independent of one another, we may calculate the combined probability of existence of real factors influencing the distribution of low densities on a chart as follows:

$$P_1 = 1 - (1 - p_1)(1 - q_1) \ldots (8).$$

In a similar way, for the high density branch we have

$$P_2 = 1 - (1 - p_2)(1 - q_2) \ldots (8').$$

The probabilities P_1, P_2 are given in table III. We may add that p_1 and p_2, as well as q_1 and q_2 are not quite independent of one another, and cannot be used therefore to calculate combined probabilities.

Table X contains average values of the probabilities of reality. These average values represent the fraction of charts which contain real irregularities of distribution, of sufficient strength to influence our criteria of irregularity. A comparison of the data for the separate galactic zones indicates a steady increase of irregularity with decreasing galactic

Table X.

Average Probabilities of Reality

	Galactic Latitude		
	± 0°... ± 22°	±23°...±44°	+ 45°...+ 87°
Total number, T . .	94	72	50
Average p_1 . .	0.78	0.37	0.00
» q_1 . .	0.84	0.41	0.02
» P_1 . .	**0.93**	**0.58**	**0.02**
» p_2 . .	0.86	0.51	0.02
» q_2 . .	0.89	0.67	0.26
» P_2 . .	**0.99**	**0.80**	**0.27**

latitude, which apparently is an effect of perspective. A more remarkable feature is the persistent prevalence of p_2, q_2, P_2, over p_1, q_1, P_1, in all galactic zones; this circumstance may be regarded as an indication of a greater importance of "clustering", compared with "obscuration", in determining the characteristics of irregularity of stellar distribution.

Tartu, December 3^d, 1933.