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On The Frequency of Proper Motions of Stars,

As Derived from The Johannesburg and Helsingfors Blink-Microscope Observations

By

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Appendix. The Luminosity and Density Laws.

By

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Abbreviations used:

T.P. = Publications de l'Observatoire Astronomique de l'Université de Tartu (Dorpat).

U.O.C. = Circular (or Circulars) of The Union Observatory, Johannesburg.

Hels. I = Recherches sur les Mouvements Propres des Étoiles dans la Zone Photographique de Helsingfors, par Ragnar Furuhjelm. I. Clichés de 9^h à 12^h. (Acta Societatis Scientiarum Fennicae. Tom. **48** No. 1. (1916).

Hels. a = Étoiles ayant un mouvement propre supérieur à $0''.5$ dans la zone photographique de Helsingfors, par Ragnar Furuhjelm. Öfversigt af Finska Vetenskaps-Societetens Förhandlingar. Bd. **59**. 1916—1917. Afd. A. No. 22.

G.P. = Publications of The Astronomical Laboratory at Groningen.

A.N. = Astronomische Nachrichten.

C.A.Z. = Cape Astrographic Zone.

C.P.D. = Cape Photographic Durchmusterung.

K. Mattiesen'i trükk, Tartus.

1. Introduction.

The methods of determining proper motions with the aid of the blink-microscope and the stereocomparator have proved most convenient from the standpoint of economy of time. The researches of Wolf with the stereocomparator, of Innes, Van Maanen, Furuhjelm with the blink-microscope have contributed much valuable material, especially in what concerns the knowledge of proper motions of faint stars. However, a sort of subjective selection depending on the size of the proper motion and, to some extent, upon the apparent magnitude also, makes the statistical use of the material found with the aid of the methods mentioned above somewhat inconvenient. In *G. P.* **30** the difficulty is overcome by simply rejecting the portion of the material which appeared to be incomplete, and by adjusting the effective area so as to take account of some residual selection in the material retained. Such a treatment leads, however, to an undue restriction of the observational data.

It is possible to avoid this restriction and to make use of a much more complete material, if the above-said selection can be determined quantitatively. A general method of investigating the selection of different sets of statistical data has been devised by the author [1]). The so-called Double-Count method, consisting in independent observations made within the same region by two or more observers, allows of determining the coefficient of perception, p, or the probable value of the observed fraction of the true number of objects. If n is the observed number, the unknown true number N is given by

$$N = \frac{n}{p}.$$

1) Über korrespondierende statistische Beobachtungen. *A. N.* B. **219** *p.* 93 (1923). Also *T. P.* **25** № 1, *pp.* 4—5, where the considerations are of quite a general character, though applied to the particular case of shooting stars.

The present investigation deals with the derivation of the coefficients of perception and the determination of the true distribution of proper motions exceeding 0″.04 annually; only the most extended blink-microscope observations, those of R. T. A. Innes, aided by H. E. Wood at Johannesburg, and of R. Furuhjelm at Helsingfors are here considered. Of these data a considerable proportion answered the conditions of the Double-Count method and allowed of determining directly the coefficients of perception; for the remaining portion of the material the coefficients of perception were estimated according to an indirect method, by using the experience gained from the directly calculated coefficients.

I am greatly indebted to Mr. H. E. Wood for communicating in Ms. some unpublished results of Dr. Innes and his own, relating to Double-Count observations in a portion of the Cape Astrographic Zone; to Dr. R. T. A. Innes for communicating some details of his method of observations with the blink; to Dr. R. Furuhjelm for supplying with unpublished details of his observations.

The data regarding proper motions of faint stars increase steadily with time; although the material used in this investigation consists of over 2000 counted objects, in a near future the material available will be much larger; if, nevertheless, it has been thought worth while to make a statistical summary of the data at present available, it is chiefly because the method here used has some bearing not only upon the statistical discussion of observations, but upon the method of the observations themselves; it may be hoped that those who work with the blink-microscope will take into account the considerations developed in this preliminary investigation, and will arrange their future work so as to permit of deriving the best statistical results. On the other hand, the material here treated is large enough to determine the general features of the distribution of stars according to proper motion and apparent magnitude and to allow of a comparison with the results of *G. P.* **30**; an independent check of our knowledge of the distribution of proper motions of faint stars may thus be obtained.

The data used for the statistical results are contained in the following publications:

U. O. C. 37, 39, 43, 46, 47, 48, 49, 53, 54, 58, containing observations chiefly by Dr. R. T. A. Innes, and partly also by H. E. Wood.

Hels. I; Hels. a., containing observations by Dr. R. Furuhjelm.

A manuscript catalogue of 1052 proper motions found by Innes and Wood in a part of *C. A. Z.* was used for studying the dependence of the coefficient of perception upon proper motion and apparent magnitude; for the statistical results these proper motions were not used; their discussion is postponed till the publication of the results for the whole zone; only proper motions of this zone exceeding 0″.50 and published in *U. O. C.* 58 were counted here.

In the computations I was aided by several members of the staff of the Tartu observatory.

2. Selection in Blink-Microscope Observations.

The character of the selection depends upon the method of observation. The method used by Furuhjelm, where the stars were examined according to a list (schematical chart) prepared in advance, leaves to determine only the selection depending upon the displacement (proper motion × interval of time) and upon the distance from the centre, whereas the selection according to apparent magnitude may be assumed to be identical with the selection of the observing list. For the observations of Innes, where the attention was not fixed at any particular star chosen in advance, the selection according to apparent magnitude enters also as an unknown factor; the inconvenience produced by this circumstance is, maybe, counterbalanced by the greater economy in the work of the observer.

The quality of the plates influences considerably the coefficients of perception; as to the distance from the centre of the plate, there is no need of taking this factor into account separately if each pair of plates is treated as a whole; the results will thus refer always to a certain effective distance from the centre.

It has been found that the coefficient of perception in searching for proper motions with the blink-microscope may generally be satisfactorily represented by the following formula:

$$p = \pi(s) \cdot \chi(m) \quad . \; . \; . \; (1),$$

where π is a function depending upon the m e a s u r e d displacement s, and χ — a function of the estimated apparent magnitude m only. Because of the effect of error-dispersion the average values of the measured quantities s and m are not identical with the averages of the true quantities; with the true quantities as arguments the functions π and χ must therefore be different from the functions here considered; it is, however, obviously a necessity to take as arguments of our functions the measured quantities.

If the same region is examined many times independently, the total coefficient of perception is given by

$$P = 1-(1-p_1)(1-p_2) \ldots (1-p_k) \ldots (2),$$

where $p_1, p_2, \ldots p_k$ are the individual coefficients of perception for each examination. Let n denote the observed number of different objects, N — the true number; then we have

$$N = \frac{n}{P}, \text{ or } N = nz \ldots (3),$$

where

$$z = \frac{1}{1-(1-p_1)(1-p_2) \ldots (1-p_k)} \ldots (4).$$

The quantity z will further be called the extrapolation factor. The effective area, σ, in the sense used in *G. P.* **30**, is given by $\sigma = PS \ldots$ (5), where S denotes the total or real area investigated, and P is defined by (2).

The individual coefficients of perception may be found from the number of common objects in the different independent examinations. Let n_1 and n_2 be the numbers of proper motions found within the same area during two independent series of observation, and let m_{12} be the number of common objects discovered independently; the coefficients of perception are given by

$$\left. \begin{aligned} p_1 &= \frac{m_{12}}{n_2}, \quad \text{and} \\ p_2 &= \frac{m_{12}}{n_1} \quad \ldots . \end{aligned} \right\} (6).$$

The weight of each value is proportional to the divisor, n_2, or n_1 respectively.

In a like manner, formula (4) of *T. P.* 25_1 (*p*.5) may serve for determining p in the case of many searches.

It may be added that as independent series of observation may serve not only observations made by different observers, but also observations made by the same observer, if care is taken to avoid the influence upon the second search of the recollection of the first search. For this purpose either the orientation may be changed, or a new pair of plates may be used, and the time between the first search and the repetition must be sufficiently long; experience by Innes and Furuhjelm indicates that such a repetition of the search by the same observer may prove to be quite independent.

The average coefficient of perception of two series of observation is given by

$$\bar{p} = \frac{2\, m_{12}}{n_1 + n_2} \quad \ldots \quad (7),$$

the denotations being those of equation (6). In the case when both searches belong to the same observer, it is advisable to use formula (7) instead of (6) and to put $p_1 = p_2 = \bar{p}$.

Of practical interest is the case when the same star lies in the field of several independent searches made by the same observer; such is the case of a star placed in overlapping portions of different pairs of plates. Let ν be the number of searches which include a certain star, and let the proper motion of this star be discovered independently in a cases, being thus missed in $\nu - a$ cases. The probable value of the coefficient of perception of the particular star is given by

$$p = \frac{a - 1}{\nu - 1}.$$

The weight of such a value of p is proportional to the number of observed cases, i. e. to

$$a\,(a - 1) + (\nu - a)\, a = a\,(\nu - 1)\ ^{1)}.$$

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1) This expression corresponds to the sum of the divisors of formula (41), p. 5, *T. P.* 25. 1, i. e. to $\sum_{i=1}^{i=k} \left(\sum_{h \neq i} n_h \right)$.

Hence the average value of the coefficient of perception for a group of stars is given by

$$\bar{p}=\frac{\sum a_i(a_i-1)}{\sum a_i(\nu_i-1)} \dots (8).$$

For $\nu=$ const. $=2$ this formula may be transformed into

$$\bar{p}=\frac{2n''}{n'+2n''} \dots (9),$$

where n'' denotes the number of stars found independently in both searches, and n' denotes the number of stars found only in one search and missed in the other. The total number of observed stars is given by $n=n'+n''$. Formula (9) is identical with (7), as $n''=m_{12}$, and $n=n_1+n_2-m_{12}$, whence $n_1+n_2=n+m_{12}=n'+2n''$.

For $\nu=$ const. $=3$ formula (8) gives

$$\bar{p}=\frac{n''+3n'''}{n'+2n''+3n'''} \dots (10), \text{ where}$$

n', n'' and n''' denote the number of stars discovered independently in one, two or all three searches respectively. The total number of different objects is given by $n=n'+n''+n'''$.

The general formula for $\nu=$ const. is

$$\bar{p}=\frac{1.2n''+2.3n'''+\dots\dots+(\nu-1)\nu n^{(\nu)}}{(\nu-1)\,[n'+2n''+3n'''+\dots\dots+\nu n^{(\nu)}]} \dots\dots (11);$$

the denotations are analogous to those of the preceding formulae.

For the study of the coefficient of perception the material presented by *C. A. Z.* may be regarded as typical. The Ms. catalogue discussed below represents the result of an examination of 95 pairs of astrographic plates by R. T. A. Innes and H. E. Wood independently. This series of measures meets thus the requirements of the Double-Count method. The list contains a total of 1052 proper motions, out of which 491 were noted by both observers quite independently, 366 were noted only by Dr. Innes and 195 only by Mr. Wood. In the following the coefficients of perception and other data relating to the two observers will be marked by the letters *i* and *w*.

Assuming formula (1), the values of the functions π and χ were determined according to a method of successive approximations similar to the method used in *T. P.* **25.** 1. pp 14—16. The results were as follows.

The *magnitude function,* χ (m), could be assumed identical for both observers; a first solution gave for this function the following values:

$\bar{m}$ (mean magn., estimated)	7.4	9.3	10.3	11.0	11.7	13.1	14.1
$\chi(m)$	0.985	0.866	0.948	0.916	0.853	0.842	0.554
Observed number	81	139	238	256	165	73	46

These values were plotted and a smooth curve drawn through the points. Finally the smoothed values contained in table 1 were adopted.

Table 1. Magnitude function in *C. A. Z.*

(the same for both observers).

m	$\leqslant$10.5	11.0	11.5	12.0	12.5	13.0	13.5	14.0	14.5	15.0
$\chi(m)$...	0.933	0.911	0.888	0.865	0.840	0.817	0.763	0.610	0.318	(0.±)

The most probable values of the function π are given in table 2.

For displacements greater than 4″ the function π attains its maximum value and becomes practically constant. Below 4″ there is a steady decrease of π; with a certain degree of approximation the variation of π for $s < 4''$ may be represented as a linear function of *log s*. Assuming this, the data were discussed anew. The new form of the coefficient of perception was written as

$$\left.\begin{array}{l} p = c\,\chi(m)\,[1 - k\,(0.6 - \log s)], \text{ for } \log s \leqslant 0.600, \text{ and} \\ p = c\,\chi(m), \text{ for } \log s > 0.600 \;\ldots\ldots\ldots\ldots \end{array}\right\} (12),$$

where $\chi(m)$ is given by table 1; c and k are coefficients to be determined for each observer separately. A least-square solution gave the following values of the coefficients:

$$I.\quad c = 0.989;\; k = 0.872;\; s_0 = 0.''284$$
$$W.\quad c = 0.998;\; k = 1.355;\; s_0 = 0.''728.$$

Table 2. The displacement-factor, $\pi(s)$, in the coefficient of perception of *C. A. Z.*

s = total measured displacement, in seconds of arc.

Some of the values of π are smoothed; in this case the original values are added in parentheses.

s	mean log s	$\pi_i(s)$	$\pi_w(s)$	Observed number
< 0″.79	−0.183	0.16	0.04	34
0″.79—0″.99	−0.042	0.254	0.204	44
1″.00—1″.25	+0.055	0.430 (0.540)	0.290 (0.378)	79
1″.26—1″.58	0.156	0.570 (0.478)	0.400 (0.380)	102
1″.59—1″.99	0.251	0.705	0.514	157
2″.00—2″.50	0.350	0.817	0.625	158
2″.51—3″.15	0.450	0.849	0.726	166
3″.16—3″.99	0.544	0.951	0.897	106
> 3″.99	> 0.601	0.998	1.058	155

Here s_0 denotes the displacement for which p becomes zero according to formula (12); it represents thus the minimum displacement which may be detected. The observed minimum displacement was: 0.″24 for *I.* and 0.″46 for *W.* Taking into account that these are measured quantities, including errors of observation, it is not surprising that they fall below the theoretical limit; on the other hand, the deviations from formula (12) appear to be real, at least partly, and for statistical purposes formula (1) used with the data of tables 1 and 2 is to be preferred. In our case we only wish to obtain a typical example of the behaviour of the coefficient of perception under real circumstances of observation, and therefore formula (12) as a convenient form of schematization will be adopted. This formula, together with the values of the coefficients quoted above, leads to the values of the coefficient of perception given in table 3. In this table P denotes the total, or effective coefficient of perception of the combined search of both observers, and is given by

$$P = 1-(1-p_i)(1-p_w) = \\ = p_i + p_w - p_i p_w,$$

in accordance with formula (2).

Besides the *C. A. Z.* there are a few series of measures by

Table 3.

Coefficients of Perception in *C.A.Z.*

s (meas. displ.)		m (estimated magnitude). $\leqslant 10.7$	11.0	11.5	12.0	12.5	13.0	13.5	14.0, 14.5
0″.5	pi	0.19	0.19	0.18	0.18	0.18	0.17	0.16	0.13
	pw	—	—	—	—	—	—	—	—
	P	**0.19**	**0.19**	**0.18**	**0.18**	**0.18**	**0.17**	**0.16**	**0.13**
0″.6	pi	0.26	0.25	0.24	0.24	0.23	0.23	0.21	0.18
	pw	—	—	—	—	—	—	—	—
	P	**0.26**	**0.25**	**0.24**	**0.24**	**0.23**	**0.23**	**0.21**	**0.18**
0″.7	pi	0.31	0.31	0.30	0.29	0.28	0.27	0.26	0.21
	pw	—	—	—	—	—	—	—	—
	P	**0.31**	**0.31**	**0.30**	**0.29**	**0.28**	**0.27**	**0.26**	**0.21**
0″.8	pi	0.36	0.35	0.34	0.33	0.32	0.31	0.29	0.23
	pw	0.05	0.05	0.05	0.04	0.04	0.04	0.04	0.04
	P	**0.39**	**0.38**	**0.37**	**0.36**	**0.35**	**0.34**	**0.32**	**0.26**
0″.9	pi	0.40	0.39	0.38	0.37	0.36	0.35	0.33	0.26
	pw	0.11	0.11	0.11	0.10	0.10	0.10	0.09	0.07
	P	**0.46**	**0.46**	**0.44**	**0.44**	**0.42**	**0.41**	**0.39**	**0.32**
1″.0	pi	0.44	0.43	0.42	0.41	0.40	0.38	0.36	0.29
	pw	0.17	0.17	0.17	0.16	0.16	0.15	0.14	0.11
	P	**0.54**	**0.52**	**0.51**	**0.50**	**0.49**	**0.48**	**0.45**	**0.37**
1″.2	pi	0.50	0.49	0.48	0.47	0.45	0.44	0.41	0.33
	pw	0.27	0.27	0.26	0.25	0.24	0.24	0.22	0.18
	P	**0.64**	**0.62**	**0.61**	**0.60**	**0.59**	**0.58**	**0.54**	**0.45**
1″.4	pi	0.56	0.54	0.53	0.52	0.50	0.48	0.46	0.36
	pw	0.36	0.35	0.34	0.33	0.32	0.31	0.29	0.23
	P	**0.71**	**0.70**	**0.70**	**0.68**	**0.66**	**0.65**	**0.62**	**0.51**
1″.6	pi	0.60	0.59	0.57	0.56	0.54	0.52	0.49	0.39
	pw	0.43	0.42	0.40	0.40	0.38	0.37	0.35	0.28
	P	**0.77**	**0.76**	**0.74**	**0.73**	**0.72**	**0.70**	**0.67**	**0.56**
1″.8	pi	0.64	0.63	0.61	0.60	0.58	0.56	0.53	0.42
	pw	0.50	0.48	0.47	0.46	0.45	0.43	0.41	0.32
	P	**0.82**	**0.81**	**0.80**	**0.78**	**0.77**	**0.75**	**0.72**	**0.61**
2″.0	pi	0.68	0.66	0.65	0.63	0.61	0.60	0.56	0.45
	pw	0.55	0.54	0.53	0.51	0.50	0.48	0.45	0.36
	P	**0.86**	**0.84**	**0.83**	**0.82**	**0.80**	**0.79**	**0.76**	**0.65**
2″.4	pi	0.74	0.73	0.71	0.69	0.68	0.65	0.61	0.49
	pw	0.65	0.63	0.62	0.60	0.58	0.57	0.54	0.43
	P	**0.91**	**0.90**	**0.89**	**0.88**	**0.87**	**0.84**	**0.82**	**0.70**
2″.8	pi	0.80	0.78	0.76	0.74	0.72	0.70	0.66	0.52
	pw	0.74	0.71	0.70	0.68	0.66	0.64	0.60	0.48
	P	**0.95**	**0.94**	**0.93**	**0.92**	**0.91**	**0.90**	**0.86**	**0.76**
3″.2	pi	0.84	0.82	0.80	0.78	0.76	0.74	0.69	0.55
	pw	0.80	0.79	0.77	0.75	0.73	0.70	0.66	0.53
	P	**0.97**	**0.96**	**0.95**	**0.95**	**0.93**	**0.93**	**0.90**	**0.79**
3″.6	pi	0.89	0.86	0.84	0.82	0.80	0.77	0.73	0.58
	pw	0.87	0.85	0.82	0.81	0.78	0.76	0.72	0.57
	P	**0.99**	**0.98**	**0.97**	**0.97**	**0.95**	**0.95**	**0.92**	**0.82**
$\geqslant$4″.0	pi	0.92	0.90	0.88	0.86	0.83	0.81	0.76	0.60
	pw	0.93	0.91	0.89	0.86	0.84	0.82	0.76	0.61
	P	**1.00**	**0.99**	**0.98**	**0.98**	**0.97**	**0.96**	**0.94**	**0.84**

Dr. Innes which permit of determining directly the coefficient of perception from overlapping portions of neighbouring pairs of plates; these series are those of *U. O. C.* **43** (Greenwich plates), **53** (Melbourne plates), and **54** (South polar region). In each case, however, the material is much smaller than in the *C. A. Z.* and does not allow of a very detailed treatment. For these series therefore only the displacement factor, $\pi(s)$, was derived independently, whereas the magnitude function was assumed as given in table 1, the m-argument of the table being changed by an amount corresponding to the difference in limiting magnitude of the given series of measures and the *C. A. Z.*

It proved difficult to determine the limiting magnitude itself; instead of this, for each list of proper motions in *U. O. C.* a standard magnitude was determined, defined by the condition that the observed percentage of proper motion stars fainter than the standard magnitude be the same as the percentage of stars fainter than estimated magnitude 11.7 in *C. A. Z.* It may be assumed that the standard magnitude thus defined corresponds to the same photographic effect as the effect produced by stars of magnitude 11.7 in *C. A. Z.* The shift in the m- argument of table 1 may thus be found; the above-mentioned percentage, about 20%, being small, the systematic error produced by the different frequency of proper motion stars of different magnitude must be negligible. For *C. A. Z.* the following percentages of proper motion stars fainter than estimated magnitude 11.7 were found:

for stars found independently by I., 19.3%;
" " " " " W., 18.4%;
for all stars (combined search of 2 observers), 22.5%.

The greater percentage for the combined search is explained by the fact that a second search adds a relatively greater number of difficult, e. g. faint objects, than of easy objects, the latter being already picked out in the first search. In determining the standard magnitude a value of the percentage was adopted between 18.4 and 22.5%, according to the character of the observations[1]). Table 4 gives a summary of the values of

1) The use of different values of the percentage is purely conventional, as the range in the standard magnitude corresponding to the extreme values of the percentage does not exceed 0.1 st. mg. and has no influence upon the slowly varying $\chi(m)$.

Table 4.

Assumed Magnitude Function, $\chi(m)$, in *U. O. C.*

The magnitudes are uncorrected, taken directly as given in the corresponding lists of proper motions.

Source	%	m_0	Magnitude Function								
U. O. C. **37,** Melbourne Cat. Plates	19.3	10.0	m	⩽ 9.2	9.5	10.0	10.5	11.0, 11.5			
			χ (m)	0.93	0.91	0.89	0.84	0.61			
U. O. C. **39,** Cape Cat. Plates	19.3	10.1	m	⩽ 8.2	8.6	9.2	9.9	10.3 to 11.5			
			χ (m)	0.93	0.92	0.89	0.84	0.61			
U. O. C. **43,** Greenwich Chart or Photometric Plates	20.0	12.4	m, Ch. or Phot.	⩽ 11.2	11.5	12.0	12.5	13.0	13.5	14.0	14.5, 15.0
			m, Cat.	⩽ 9.7	10.0	10.5	11.0	11.5	12.0	12.5	13.0, 13.5
„ „ Greenwich Cat. Plates	20.0	11.1	χ (m)	0.93	0.93	0.90	0.87	0.85	0.83	0.78	0.61
U. O. C. **46,** Cluster Regions	18.4	13.7	m	⩽ 12.4	13.0	14.0	15.0				
			χ (m)	0.93	0.91	0.84	0.61				
U. O. C. **47,** Paris (1 pair)	19.3	15.1	m	⩽ 13.0	14.0	15.0					
			χ (m)	0.93	0.90	0.80					
U. O. C. **48,** Sydney Ch. Pl.	20.0	11.6	m	⩽ 10.4	11.0	12.0	13.0	14.0			
			χ (m)	0.93	0.91	0.86	0.82	0.61			
U. O. C. **49,** Algiers Cat. Pl.	19.3	11.5	m	⩽ 10.9	11.2	12.2					
			χ (m)	0.93	0.85	0.56					
U. O. C. **53,** Melbourne Ch. Pl.	21.0	12.1	m	⩽ 11.2	11.5	12.0	12.5	13.0	13.5	14.0	14.5
			χ (m)	0.93	0.91	0.89	0.86	0.84	0.82	0.76	0.61
U. O. C. **54,** Cape South Pol.	22.5	10.1	m	⩽ 9.7	10.0	10.5	11.0				
			χ (m)	0.93	0.86	0.82	0.61				

standard magnitude (m_0) derived in this way, together with the adopted values of the magnitude function. In some cases the peculiarities of the magnitude scale used (e. g. of *C. P. D.)* were taken into account in determining the magnitude shift for values of m different from those of the standard magnitude.

Assuming the values of χ as given in table 4, the displacement-factor, π, was found for the overlapping portions of plates of *U. O. C.* **43** and **53** according to formula (9), and for *U. O. C.* **54** according to formula (8), where it must be put

$$\overline{p} = \pi(s) . \overline{\chi}(m),$$

$\overline{\chi}(m)$ denoting the average value of the magnitude function for the considered group of stars. Table 5 contains the results.

Table 5.

Values of π (s), derived from overlapping portions of plates.

1) *U.O.C.* **43.**

s	$\leqslant 1''.54$	$1''.55...1''.95$	$2''.0...2''.9$	$3''.0...3''.9$	$\geqslant 4''.0$
mean log s	+0.080	0.238	0.378	0.529	$\geqslant$ 0.602
mean χ (m)	0.926	0.888	0.906	0.912	0.905
π (s)	0.17	0.17	0.43	0.76	0.89
Observed number of stars	12	12	29	15	25

2) *U.O.C.* **53.**

s	$0''.62...0''.80$	$0''.81...1''.06$	$1''.07...1''.44$	$1''.45...1''.92$	$1''.93...2''.56$	$2''.57...3''.40$	$3''.41...4''.00$	$\geqslant 4''.01$
mean log s	−0.152	−0.030	+0.096	0.224	0.348	0.472	0.568	>0.602
mean χ(m)	0.919	0.896	0.912	0.909	0.911	0.918	0.908	0.918
π (s)	0.44	0.20	0.49	0.59	0.66	0.88	0.84	1.02
Observed number of stars	8	10	24	44	53	44	13	34

3) *U.O.C.* **54.**

s	$\leqslant 0''.84$	$0''.85...1''.06$	$1''.07...1''.34$	$1''.35...1''.69$	$1''.70...2''.13$	$\geqslant 2''.14$
mean log s	−0.32	−0.01	+0.07	0.16	0.27	>0.304
π (s)	0.11	0.13	0.21	0.45	0.56	0.97
Observed number of stars[1])	4	9	5	7	6	7

1) Taking into account the great number of overlaps, each observed star of *U. O. C.* **54** possesses, in the determination of π, a weight about twice as great as the weight of a star in *U. O. C.* **43** or **53.**

The plates of the Helsingfors Astrographic Zone investigated by Dr. R. Furuhjelm cover each point of the zone at least twice, and a part of the zone is covered three or more times; only at the northern and southern boundaries two bands about 1^0 wide were covered once. As each pair of plates was examined by Furuhjelm quite independently, the observations presented a large material for determining the coefficients of perception. As stated at the beginning of this section, for the Furuhjelm stars the selection according to magnitude may be neglected; perhaps the faintest stars present an exception. We may thus put $\chi(m) = 1$ and $p = \pi$.

The arrangement of observations by Furuhjelm is, however, such that our theoretical formulas cannot be applied without modification. Although the search may be regarded as made according to the rules of the Double Count, in the measures and the final catalogue some of the stars found were not included; thus the effective coefficient of perception of the catalogue is not the same as the original coefficient of perception of the search. This artifical diminution of the coefficient of perception must be taken into account.

The method of the search is described by Furuhjelm in *Hels. I*, p. 7. During the search some of the stars were noted as showing certainly a displacement, whilst for others the displacement appeared to be doubtful; in the latter case the stars were marked with a ?. Each star which on a pair of plates appeared to have a certain displacement was measured, whereas the stars marked with a ? were only measured if found independently upon, at least, two pairs of plates. Stars noted once with a ? were measured only at the borders of the zone covered by a single pair of plates; on the remaining parts of the zone these stars were rejected. There are no data for the rejected stars, as they were not measured.

For the stars of his catalogue Dr. Furuhjelm computed, according to formula (9)[1]) or (10)[2]), two kinds of values of the apparent coefficient of perception, p' and p''; the first value by counting the discoveries marked by a ? , the second — by ignoring these discoveries. Table 6 gives these quantities, together

1) For regions cowered twice.
2) For regions cowered thrice.

with p_0, the coefficient of perception of the original search on a single pair of plates, and P, the effective coefficient of perception of the catalogue. The values of p_0 and P were computed from p' and p'' according to formulae given below. Only stars brighter than magnitude 10.6 on the Helsingfors scale were used in deriving the data of the table, as stars of magnitude 10.6 or fainter do not occur on all pairs of plates. The table is arranged according to the annual proper motion (μ); the effective displacement is obtained by multiplying with the average interval = 18.33 years.

Table 6.

Coefficients of Perception for the Furuhjelm Stars.

μ	0″.051... ...0″.060	0″.061... ...0″.070	0″.071... ...0″.080	0″.081... ...0″.090	0″.091... ...0″.100	0″.101... ...0.″110	>0″.110
mean *log s*	+0.007	0.079	0.141	0.195	0.243	0.286	>0.304
1) Regions covered twice							
p'	0.644	0.677	0.792	0.732	0.865	0.833	0.989
p''	0.263	0.340	0.500	0.545	0.733	0.733	0.977
p_0	0.492	0.559	0.724	0.684	0.847	0.819	0.987
P	**0.509**	**0.614**	**0.802**	**0.815**	**0.944**	**0.927**	**1.000**
2) Regions covered by 3 pairs of plates.							
p'	0.364	0.632	0.647	0.791	0.889	0.682	0.969
p''	0.136	0.391	0.480	0.533	0.778	0.286	0.885
p_0	0.263	0.597	0.626	0.779	0.887	0.631	0.968
P	**0.393**	**0.835**	**0.887**	**0.953**	**0.995**	**0.809**	**1.000**

Let p_1 and p_2 denote the probabilities that during one search a star of a given proper motion will be found with certainty, or will be marked with a ? respectively. The coefficient of perception of the search is then given by

$$p_0 = p_1 + p_2 \quad . \; . \; . \quad (13).$$

During a search the following events may happen:

1) a proper motion may be noted with certainty; this event we shall denote by the symbol +; its probability is p_1;

2) a proper motion may be marked with a ?; the ? will be used as the symbol of this case; the probability is p_2;

3) a proper motion may be missed, in what case a — will be used as symbol; the probability is $1 - p_0 = 1 - (p_1 + p_2)$.

Let us first consider the case of a region covered by two pairs of plates. Let N denote the true number of proper motions in the region. The different combinations and their probable frequency will be as follows:

	Combination of events	Probable frequency	Use made of in the catalogue
(a)	+ +	$N p_1^2$	included
(b)	+ ? ? +	$2 N p_1 p_2$	"
(c)	? ?	$N p_2^2$	"
(d)	+ − − +	$2 N p_1 [1-(p_1+p_2)]$	"
(e)	? − − ?	$2 N p_2 [1-(p_1+p_2)]$	rejected
(f)	− −	$N [1-(p_1+p_2)]^2$	undiscovered

The effective coefficient of perception of Furuhjelm's catalogue is given by

$$P=\frac{(a)+(b)+(c)+(d)}{N}, \text{ which gives}$$

$$P=p_0^2+2p_1(1-p_0) \ldots (14),$$

p_0 being defined by formula (13).

The apparent coefficient of perception p', obtained by counting the ?, is given by formula (9), where $n''=(a)+(b)+(c)$, and $n'=(d)$. The apparent coefficient of perception p'', obtained by ignoring the ?, is given by formula (9) with $n''=(a)$ and $n'=(b)+(d)$. Thus,

$$p'=\frac{p_0^2}{p_0^2+p_1(1-p_0)} \ldots (15), \text{ and}$$

$$p''=p_1 \ldots (16).$$

Formulae (16), (15), (13) and (14) furnish a means for computing p_0 and P, if p' and p'' are given.

In quite a similar way, for the case of a region covered by three pairs of plates, the following formulae may be derived:

$$P=p_0^3+3p_0^2(1-p_0)+3p_1(1-p_0)^2 \ldots (17);$$

$$p'=\frac{p_0^3+p_0^2(1-p_0)}{p_0^3+2p_0^2(1-p_0)+p_1(1-p_0)^2} \ldots (18), \text{ which}$$

may be transformed into

$$2-p_0+p_1\left(\frac{1}{p_0}-1\right)^2=\frac{1}{p'} \;\ldots\ldots (18'); \text{ and}$$

$$p''=p_1 \;\ldots\ldots \text{identical with formula (16).}$$

The different values of the displacement factor, $\pi(s)$, determined directly and contained in tables 2, 5 and 6, are plotted on fig. 1; for each observational series a smooth curve is drawn which satisfies best the individual points. Instead of the values

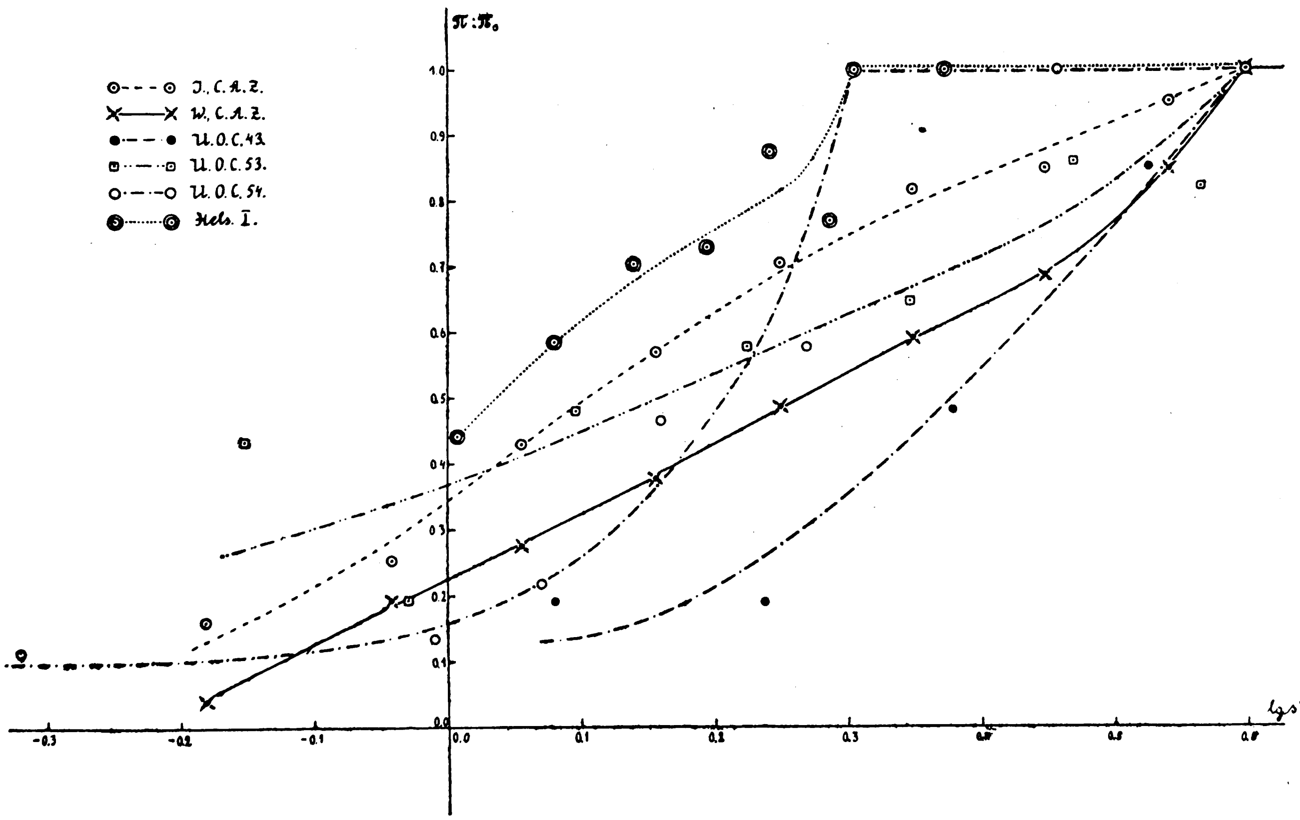

Fig. 1.

of π themselves the ratio $\pi:\pi_0$ is used, where π_0 denotes the maximum value of π which is attained for great values of s. As in all series π_0 little differs from 1, the curves represent with sufficient approximation also the absolute values of π.

The variety of the curves suggests a great variety in the observing conditions, depending upon the method of observation, the individuality of the observer, and the quality of the plates. A considerable influence upon the shape of the curves must exert the error-dispersion of the measured displacements; the same measured value may correspond to different *mean true* values, according to the accuracy of the measures and the frequency-function of the observed displacements.

Notwithstanding the variety, some features common to all curves may be noted. There exists such a value $s = S$ that for $s > S$, π becomes constant and equal to π_0 ($\pi_0 = 1$ approximately); for $s < S$, π decreases steadily, and it may be assumed that for a certain $s = s_0$, the value $\pi = 0$ is reached. For the majority of the curves between s_0 and S, π may be represented roughly as a linear function of *log s;* certain exceptions present *U. O. C.* **43** (Greenwich overlapping regions) and *U. O. C.* **54** (Cape South Polar plates).

If a few points at the lower portion of the curve are given, the upper portion of the curve may be drawn with little uncertainty, the uncertainty attaining on the average not more than 5%. This circumstance may help to determine π in cases where no direct determinations are available. For this purpose the distribution of the smallest measured displacements may be used. For a given method of searching it may be assumed that the number of small displacements does not depend upon the real distribution of displacements, but upon the coefficient of perception only. Thus we may assume that

$$\frac{\sum\limits_{s=s_0}^{s=s_1} n}{N} = f\left[\frac{p(s_1)}{p_0}\right] \quad . \quad . \quad . \quad ,$$

or that

$$\frac{p(s_1)}{p_0} = F\left[\frac{\sum\limits_{s=s_0}^{s=s_1} n}{N}\right] \quad . \quad . \quad . \qquad (19),$$

where

$$\sum_{s=s_0}^{s=s_1} n$$

denotes the observed number of displacements from the smallest, $s = s_0$, to $s = s_1$, N — the total observed number, p — the coefficient of perception, and p_0 — the coefficient of perception for $s > S$. The character of the function F may be found from the series with directly determined coefficients of perception. It may be remarked that for a *single search* it may be assumed, with sufficient approximation,

$$\frac{p(s_1)}{p_0} = \frac{\pi(s_1)}{\pi_0} \quad . \quad . \quad . \qquad (20).$$

As a standard for determining the function F the *C. A. Z.* was assumed, because it represents the most extended series with directly determined coefficients of perception. The data used as the standard of comparison are contained in table 7. Section a) of this table may be used in the case of a search repeated twice, sections b) and c) relating to searches made only once. In fact, only sections a) and b) of table 7 were used, as all series which required the application of the indirect method belonged either to R. T. A. Innes, or were repeated twice.

Table 7.

C. A. Z.

a) Combined search of two observers.

Σn	40	80	120	160	200
$\Sigma n : N$	0.038	0.076	0.114	0.152	0.190
s_1	0″.78	0″.98	1″.13	1″.22	1″.37
[1]) $P : P_0$ . . .	0.35	0.52	0.60	0.64	0.70

b) Proper motions found by *I.* only.

$\Sigma n : N$	0.038	0.076	0.114	0.152	0.190
s_1	0″.78	1″.00	1″.17	1″.28	1″.43
$p : p_0$	0.21	0.34	0.45	0.51	0.58

c) Proper motions found by *W.* only.

$\Sigma n : N$	0.038	0.076	0.114	0.152	0.190
s_1	0″.96	1″.16	1″.31	1″.47	1″.59
$p : p_0$	0.22	0.30	0.36	0.41	0.44

Table 8 contains a comparison of the two methods of determining the coefficient of perception; from this table it may be inferred that the indirect method leads to fairly reliable results.

Directly determined coefficients of perception were used in the statistical discussion for the Furuhjelm stars, and of the

1) P is the effective coefficient of perception of the combined search.

Table 8.

Comparison of the directly and indirectly determined coefficients of perception.

U. O. C. **43**, overlapping regions; searched twice.

	s	1″.70	2″.10	2″.28	2″.31	2″.34
[1] p	p directly determined	0.36	0.54	0.64	0.65	0.65
	indirectly "	0.35	0.52	0.60	0.65	0.70

U. O. C. **53**, overlapping regions; searched twice.

	s	0″.84	1″.01	1″.17	1″.37	1″.43
[1] p	directly determined	0.45	0.54	0.62	0.66	0.68
	indirectly "	0.35	0.52	0.60	0.65	0.70

Johannesburg proper motions only for those contained in *U. O. C.* **54**. For the remaining *U. O. C.* coefficients of perception determined indirectly according to the method described above were assumed. Although for *U. O. C.* **43** and **53** coefficients of perception for the overlapping regions were determined, these coefficients relating to the marginal portions of the plates cannot be regarded as representative of the whole zone; in these cases the coefficients of perception for the not overlapping regions were determined indirectly, and for the whole zone the weighted mean of these indirect values and the directly determined coefficients of the marginal regions was assumed.

3. Separate Results.

The magnitudes are *photographic*, reduced to the Harvard scale. The systematic corrections for the faintest stars, for which only estimated magnitudes were available, could not be determined with precision; thus some uncertainty is introduced in the result; the uncertainty is, however, not considerable.

In *G. P.* **30** results of photographic observations are reduced to the Harvard visual scale with the aid of corrections derived for stars as a whole; such a proceeding is not quite legitimate, because the average colour of stars of a given magnitude varies systematically with the proper motion, stars of large

1) Average for all magnitudes.

proper motion being considerably redder than the average; though the difference is not very great, it is quite sensible, attaining as great a value as 0.5 magnitudes. In our discussion it appeared, therefore, preferable to represent results of photographic observations by a photographic scale of magnitudes, instead of reducing them to a *pseudo-visual* scale like the scale of *G. P.* **30.**

In the material discussed a considerable portion of the magnitudes in the *U. O. C.* were taken from *C. P. D.*; in cases where the magnitudes were not already reduced to the Harvard scale, the following average corrections were used:

magn., *C. P. D.*	7.0	8.0	8.5	9.0	9.5	10.0	10.5	11.0	(12.0)	(13.0)
Correction . . .	0.00	+0.41	+0.62	+0.88	+1.15	+1.35	+1.53	+1.71	(+2.07)	(+2.43)

The values given in parentheses are extrapolated; these corrections were used in the case when magnitudes of faint stars were given by Innes on a „supposed continuation of the *C. P. D.* scale“.

In some cases star counts were used for determining the scale of magnitudes in accordance with table IV of *G. P.* **27.**

The most important systematic correction of the statistical data consists in freeing the results from the effect of error-dispersion. The effect of error-dispersion both upon the proper motions and upon the magnitudes was taken into account. The correction for error-dispersion was attained by a numerical method of successive approximations; Eddington's simple formula[1]) could not be used, as neither the probable error, nor the curvature of certain parts of the observed frequency function was small, and, moreover, in certain cases it appeared necessary to assume a variable value of the probable error[2]).

The correction for error-dispersion in magnitudes of the stars of *C. P. D.* was assumed equal to the difference of the correction of the proper motion stars and the stars as a whole. This was made because the correction for error-dispersion is already included in the systematic corrections of *C. P. D.*

1) *Monthly Notices,* **73.** p. 359.

2) Were the magnitude scale of the proper motion stars determined from a comparison with direct photometric measures of the same stars, no correction for dispersion in the magnitudes would be needed; such was, howewer, not the case.

The displacements are ordinarily measured in two coordinates; it is natural to assume a Gaussian distribution for the two Decartian components; in this case the distribution of errors of the total displacement will be not a Gaussian, but a certain skew curve, of the form

$$s\, e^{-h^2(s+s_0)^2}\, ds \int_{\alpha=0}^{\alpha=\pi} e^{2h^2 s s_0 \cos\alpha}\, d\alpha\,;$$

the distribution approaches a Gaussian only when s_0 is large as compared with the probable error. If e_x is the standard deviation of a component, e_s — the standard deviation of the total displacement, the ratio $e_s : e_x$ changes from 1.33 for $s_0 : e_x = 0$ to 1.00 for $s_0 : e_x = \infty$.

Now, the method of blink-microscope observations eliminates automatically values of s near zero; it therefore appeared quite permissible to assume a Gaussian distribution for the s also, with a constant effective probable error $=$ 1.1 times the probable error in one coordinate. The numerical value of the probable error could be derived from repeated measures of the same star (overlapping plates, etc.); for certain *U. O. C.* where the p. e. could not be derived directly, it was estimated by analogy with other similar series of measures.

The probable error of one estimated magnitude was assumed equal to ± 0.38 st. mg., a value derived from the data of *U. O. C.* **53.**

The probable error of the *C. P. D.* magnitudes was assumed equal to ± 0.18 st. mg. In *Harvard Annals* **76.** 12 the probable error is found as ± 0.24, but allowing for the p. e. in the photometric magnitudes, in the spectra and in the assumed colour-indices of the comparison stars, the value assumed above may be obtained.

The unsmoothed observed frequency-functions were used in deriving the corrections for error-dispersion. The correction of the distribution of displacements (proper motions) was assumed the same for all magnitudes, and the correction of the distribution of magnitudes — the same for all proper motions. Such a way of treatment is not quite correct from the theoretical standpoint because the distribution of magnitudes changes systematically with the proper motion, and vice versa; but, as a first approximation, our simplification of the problem may be regarded as satisfactory. When the magnitude scale was derived from star-counts, the latter were also corrected for error-dispersion.

According to formula (3), the true number is given by

$$N = n\ z, \text{ where}$$

$$z = \frac{1}{P}.$$

When stars are counted within certain limits of proper motion and magnitude, the values of P for the individual stars are not the same; therefore the mean value of z is not equal to the inverse of the average P. The value which we need is the average of the extrapolation factor, z. To satisfy this condition, for each star in *U. O. C.* the probable value of z was determined individually; in counting, the sum of the individual z was taken. For the Furuhjelm stars average values of P were used for groups of stars, but the value of z was corrected for the dispersion of P within the group.

In the following tables data are given only for limits of proper motion, or magnitude, for which a reliable estimate of the coefficient of perception could be made; in several cases the faintest category of magnitude was rejected because of the unknown limiting magnitude; the effective limiting magnitude cannot always be assumed the same for the proper motion stars and the stars as a whole. The rejection of small proper motions, or of magnitudes took place only after making the correction for error-dispersion; it is evident that this correction can be made only by using the entire data of observation.

Fractional observed numbers are chiefly the result of introducing the correction for error-dispersion.

The following tables contain a summary of the counts made from the different lists of proper motions. The denotation of the magnitudes is anologous to the denotation used in *G. P.* **30**; thus magnitude 10 means the limits from 9.45 to 10.44, photographic H. S.

The limits of proper motion are chosen so that they represent approximately a geometrical progression.

The observed number is denoted by n; N is the most probable value of the true number. The ratio $N:n$ gives the average value of the extrapolation factor z. The area covered by the

Table 9.

U. O. C. 37 (Melbourne Plates).

The proper motions found in the zone —66° are not counted, as they are included in *U. O. C.* **53.**

Once searched by R. T. A. Innes. Average interval = 19 years.

Effective coefficient of perception $p = \pi(s) \cdot \chi(m)$.

Assumed $\pi(s)$, determined indirectly:

s	1″.0	1″.2	1″.4	1″.6	1″.8	2″.0	2″.4	2″.8	3″.2	3″.6	4″.0
$\pi(s)$	0.19	0.30	0.42	0.54	0.63	0.70	0.80	0.87	0.92	0.96	0.98

For $\chi(m)$ consult table 4.

Corrected for error-dispersion with

a probable error in s assumed = $\pm 0''.48$, and

probable error in magnitudes: *m, H.S.*	⩽ 10	11	12	13
p. e.	±0.18	±0.26	±0.34	±0.38

a) Galactic latitude 0°...20°; area = 31□°

Centennial *p. m.*		Photographic magnitude, *H.S.* ⩽6.44	7	8	9	10	11
12″.5...16″.4	*n*	0	0	0.8	3.1	3.3	3.1
	N			**1.2**	**5.3**	**4.5**	**4.7**
16″.5...21″.4	*n*	0	0	0	0	1.7	1.1
	N					**2.2**	**1.5**
21″.5...28″.4	*n*	0	0	0	1.0	0	0.1
	N				**1.1**		**0.1**

b) Galactic latitude 20°...40°; area = 14□°

Centennial *p. m.*		Photographic magnitude, *H.S.* ⩽6.44	7	8	9	10	11
12″.5...16″.4	*n*	0	1.1	1.6	0.3	0.5	0.8
	N		**1.4**	**2.1**	**0.3**	**0.9**	**1.2**
16″.5...21″.4	*n*	0	0.8	1.2	0.5	0.3	0.3
	N		**1.0**	**1.4**	**0.6**	**0.3**	**0.3**
21″.5...28″.4	*n*	0	0	0	0.2	0.6	0
	N				**0.2**	**0.7**	

c) Galactic latitude 40°...90°; area = 19□°

Centennial *p. m.*		Photographic magnitude, *H.S.* ⩽6.44	7	8	9	10	11
12″.5...16″.4	*n*	0	0	0.6	0.8	2.0	1.3
	N			**1.0**	**1.1**	**3.2**	**1.9**
16″.5...21″.4	*n*	0	0	0	0	0.9	1.1
	N					**1.0**	**1.3**
21″.5...28″.4	*n*	0	0	0	0	0.3	1.0
	N					**0.3**	**1.2**

d) Galactic latitude 0°...90°; area = 64□°

Centennial *p. m.*		Photographic magnitude, *H.S.* ⩽6.44	7	8	9	10	11
28″.5...37″.4	*n*	0	0	0	0	0.1	0
	N					**0.1**	
37″.5...49″.4	*n*	0	0	1.0	0	0.2	0.8
	N			**1.1**		**0.2**	**0.9**
49″.5...146″.4	*n*	0	0	0	0	0	0
	N						
146″.5...194″.4	*n*	1.0	0	0	0	0	0
	N	**1.1**					
⩾194″.5	*n*	0	0	0	0	0	0.8
	N						**1.0**

Table 10.

U. O. C. 39 (Cape Plates).

Proper motions above 49″.4 per century are not counted, as they are included in *U. O. C.* 58.

Once searched by R. T. A. Innes. Average interval = 21.5 years.

Effective coefficient of perception $p = \pi(s) \cdot \chi(m)$.

Assumed $\pi(s)$, determined indirectly:

$s =$	1″.0	1″.2	1″.4	1″.6	1″.8	2″.0	2″.4	2″.8	3″.2	3″.6	4″.0
$\pi =$	(0.10)	(0.12)	0.15	0.23	0.38	0.50	0.67	0.79	0.87	0.92	0.96

For $\chi(m)$ consult table 4.

Corrected for error-dispersion with

probable error in $s = \pm 0''.28$;

probable error in m: magn., *H.S.* = $\leqslant 10$ 11 12

p. e. = ± 0.18 ± 0.21 ± 0.33.

a) Gal. lat. 0°...20°; area = 94□°

Centennial p. m.		Photogr. magn., H. S. ⩽6.44	7	8	9	10	11
9″.5...12″.4	n	0.3	0.3	2.2	1.9	3.2	14.5
	N	**0.6**	**0.6**	**4.0**	**3.4**	**6.4**	**30.4**
12″.5...16″.4	n	0	1.4	2.1	4.8	5.1	11.6
	N		**1.7**	**2.8**	**6.1**	**6.9**	**19.3**
16″.5...21″.4	n	0	1.7	1.3	2.8	1.9	4.5
	N		**1.9**	**1.5**	**3.2**	**2.2**	**6.2**
21″.5...28″.4	n	0	1.2	0	0.3	2.6	7.1
	N		**1.4**		**0.4**	**3.0**	**10.2**

b) Gal. lat. 20°...40°; area = 35□°

Centennial p. m.		Photogr. magn., H. S. ⩽6.44	7	8	9	10	11
9″.5...12″.4	n	0	0.4	0.1	1.2	1.3	4.7
	N		**1.0**	**0.2**	**2.2**	**2.3**	**10.3**
12″.5...16″.4	n	0.1	0.2	0.9	0.1	0.4	3.1
	N	**0.1**	**0.3**	**1.1**	**0.1**	**0.5**	**4.8**
16″.5...21″.4	n	0.7	0	0.1	0	0	2.5
	N	**0.8**		**0.1**			**3.3**
21″.5...28″.4	n	0.2	1.0	0	0	0	2.0
	N	**0.2**	**1.1**				**2.9**

c) Gal. lat. 40°...90°; area = 45□°

Centennial p. m.		Photogr. magn., H. S. ⩽6.44	7	8	9	10	11
9″.5...12″.4	n	0	0.1	0.8	0.9	3.1	5.9
	N		**0.1**	**1.3**	**2.2**	**5.6**	**13.4**
12″.5...16″.4	n	0	0	0.4	2.0	4.0	6.9
	N			**0.5**	**2.7**	**5.8**	**10.3**
16″.5...21″.4	n	0	0	0.1	0.9	1.3	4.2
	N			**0.1**	**1.0**	**1.6**	**5.4**
21″.5...28″.4	n	0	0	0.2	1.4	0.9	3.0
	N			**0.2**	**1.5**	**1.1**	**4.6**

d) Gal. lat. 0°...90°; area = 174□°

Centennial p. m.		Photogr. magn., H. S. ⩽6.44	7	8	9	10	11
28″.5...37″.4	n	0	0.3	2.9	1.8	0	4.0
	N		**0.3**	**3.3**	**2.1**		**5.4**
37″.5...49″.4	n	0	2.1	1.1	1.9	0.9	3.0
	N		**2.3**	**1.2**	**2.2**	**1.1**	**4.2**

Table 11.

U. O. C. 43 (Greenwich plates).

Searched once by R. T. A. Innes. Average interval = 19.5 years.

Effective coefficient of perception for the whole zone $p = \pi(s) \cdot \chi(m)$, derived partly indirectly, partly directly.

Assumed values of $\pi(s)$:

	$s =$	0″.7	0″.8	0″.9	1″.0	1″.2	1″.4	1″.6	1″.8	2″.0	2″.4	2″.8	3″.2	3″.6	4″.0
for not overlapping	$\pi =$	0.11	0.13	0.18	0.22	0.31	0.42	0.55	0.61	0.68	0.79	0.85	0.90	0.94	0.96
for overlapping [1]	$\pi =$	0.13	0.14	0.16	0.18	0.23	0.30	0.36	0.46	0.56	0.74	0.96	1.07	1.09	1.10
assumed effective π, whole zone		0.11	0.13	0.18	0.21	0.30	0.40	0.52	0.58	0.66	0.78	0.87	0.93	0.97	0.98

$\chi(m)$ is given by table 4.

Corrected for error-dispersion with

$$p.\ e.\ \text{in}\ s = \pm 0''.50;$$

p. e. in *m*: magn., *H. S.* =	$\geqslant 12$	13	14
p. e. =	± 0.10	± 0.24	± 0.37

a) Gal. lat. 0°...20°; area = 144 □° to 12 *H. S.*, and 139 □° for 13 *H. S.*

Centennial *p.m.*		Photogr. magn., *H. S.* $\leqslant 6.44$	7	8	9	10	11	12	13
9″.5...12″.4	*n*	0.7	1.7	1.3	7.3	10.7	10.5	5.7	8.5
	N	**1.0**	**3.1**	**3.5**	**14.0**	**20.3**	**19.5**	**10.6**	**16.3**
12″.5...16″.4	*n*	0.2	1.7	0.5	6.7	8.1	11.2	4.1	8.5
	N	**0.2**	**2.3**	**0.7**	**9.7**	**11.5**	**15.4**	**5.8**	**12.1**
16″.5...21″.4	*n*	0	1.3	0.5	4.8	4.6	5.5	1.8	5.3
	N		**1.4**	**0.6**	**5.4**	**5.4**	**6.4**	**2.0**	**6.6**
21″.5...28″.4	*n*	0	0	0	2.4	2.3	1.6	0.5	1.7
	N				**2.6**	**2.6**	**1.8**	**0.5**	**2.0**

b) Gal. lat. 20°...40°; area = 68 □° to 12 *H. S.*, and 67 □° for 13 *H. S.*

Centennial *p.m.*		Photogr. magn., *H. S.* $\leqslant 6.44$	7	8	9	10	11	12	13
9″.5...12″.4	*n*	0	0	1.7	1.3	2.8	3.6	4.5	6.4
	N			**3.3**	**2.5**	**4.9**	**7.1**	**8.2**	**11.7**
12″.5...16″.4	*n*	0	0	0.8	0.5	2.4	2.2	3.5	3.6
	N			**1.2**	**1.0**	**3.3**	**3.0**	**5.3**	**5.4**
16″.5...21″.4	*n*	0.5	0	0.1	0.2	0.9	1.9	1.2	2.0
	N	**0.5**		**0.1**	**0.3**	**1.1**	**2.2**	**1.4**	**2.8**
21″.5...28″.4	*n*	0.5	0	0	1.0	0.1	1.2	0.3	0.2
	N	**0.5**			**1.1**	**0.1**	**1.2**	**0.4**	**0.2**

1) Searched twice.

Table. 11. Continued.

Centennial *p.m.*		Photogr. magn., *H. S.* ⩽6.44	7	8	9	10	11	12	13
		c) Gal. lat. 40°...90°; area = 97 □° to 12 *H. S.*, and 95 □° for 13 *H. S.*							
9″.5...12″.4	*n*	0.3	1.1	2.0	6.0	7.8	9.4	3.5	6.3
	N	**0.6**	**1.8**	**3.5**	**12.5**	**14.8**	**16.7**	**7.7**	**14.6**
12″.5...16″.4	*n*	0.1	1.0	1.2	4.5	5.8	6.9	4.5	5.4
	N	**0.1**	**1.4**	**1.6**	**6.6**	**8.4**	**9.7**	**7.1**	**9.7**
16″.5...21″.4	*n*	0	0.5	0.6	3.3	2.7	4.7	3.0	3.7
	N		**0.6**	**0.7**	**3.9**	**3.4**	**5.3**	**3.7**	**4.7**
21″.5...28″.4	*n*	0	0	0	2.6	2.1	0.7	1.0	1.1
	N				**2.9**	**2.3**	**0.7**	**1.1**	**1.3**

Centennial *p.m.*		Photogr. magn., *H. S.* ⩽6.44	7	8	9	10	11	12	13
		d) Gal. lat. 0°...90°; area = 309 □° to 12 *H. S.*, and 301 □° for 13 *H. S.*							
28″.5...37″.4	*n*	0	1.0	2.0	2.0	4.0	2.0	1.5	4.2
	N		**1.1**	**2.2**	**2.1**	**4.3**	**2.1**	**1.7**	**5.3**
37″.5...49″.4	*n*	0	0	1.0	1.0	1.0	0.1	1.4	2.3
	N			**1.0**	**1.1**	**1.1**	**0.1**	**1.6**	**2.5**
49″.5...65″.4	*n*	1.0	0	1.0	1.0	0	0	0.1	0.8
	N	**1.0**		**1.1**	**1.0**			**0.1**	**1.0**
65″.5...86″.4	*n*	0	0	1.0	0	0	0	0	0
	N			**1.1**					
86″.5...146″.4	*n*	0	0	0	0	0	0	0	0
	N								
146″.5...194″.4	*n*	0	0	0	0	1.0	0.1	0.6	0.3
	N					**1.1**	**0.1**	**0.6**	**0.3**
⩾194″.5	*n*	0	0	0	0	1.0	0	0	0
	N					**1.0**			

Table 12.

U. O. C. 46 (Southern Cluster Regions).

Examined twice, once by Innes and once by Wood; η Argus region examined three times. Average Interval 23 years.

Effective coefficient of perception

a) for η Argus region $p = 1-[1-\pi(s) . \chi(m)]^3$;

b) for the remaining regions $p = 1-[1-\pi(s) . \chi(m)]^2$.

Assumed values of $\pi(s)$, determined indirectly:

$s =$	0″.5	0″.6	0″.7	0″.8	0″.9	1″.0	1″.2	1″.4	1″.6	1″.8	2″.0	2″.4	2″.8	3″.2	$\geqslant$ 3″.6
$\pi =$	0.05	0.13	0.22	0.30	0.37	0.42	0.51	0.60	0.67	0.73	0.78	0.86	0.92	0.96	1.00

$\chi(m)$ given in table 4.

The probable error in s, $\pm 0''.21$, being small, the proper motions were not corrected for error-dispersion.

The distribution of magnitudes was corrected for error-dispersion, assuming the following values of the probable error:

magn., *H. S.*	$\leqslant$10	11	12	$\geqslant$13
p. e.	$\pm$0.18	$\pm$0.21	$\pm$0.24	$\pm$0.30

All regions fall within the galactic zone 0°...20°. Area = 29□°.

Centennial *p. m.*		Photographic magnitude, *H. S.* $\leqslant$6.44	7	8	9	10	11	12	13
4″.0...5″.4	*n*	1.0	0	1.0	0	4.0	5.0	10.0	18.4
	N	**1.5**		**1.2**		**5.9**	**6.2**	**14.5**	**25.6**
5″.5...7″.4	*n*	0	0	1.0	2.0	7.0	4.0	7.0	12.4
	N			**1.3**	**2.2**	**8.4**	**5.1**	**8.0**	**14.7**
7″.5...9″.4	*n*	0	0	0	1.0	7.0	6.0	5.0	7.6
	N				**1.1**	**7.9**	**6.6**	**5.6**	**8.5**
9″.5...12″.4	*n*	0	0	1.0	0	1.0	5.0	3.0	6.0
	N			**1.0**		**1.0**	**5.2**	**3.2**	**6.3**
12″.5...16″.4	*n*	1.0	1.0	1.0	0	1.0	3.0	2.0	2.6
	N	**1.0**	**1.0**	**1.0**		**1.0**	**3.0**	**2.0**	**2.7**
16″.5...21″.4	*n*	0	1.0	1.0	0	1.0	0	4.0	0
	N		**1.0**	**1.0**		**1.0**		**4.0**	
21″.5...28″.4	*n*	0	0	0	0	0	0	0	1.0
	N								**1.0**
28″.5...37″.4	*n*	0	0	1.0	2.0	0	0	0	0
	N			**1.0**	**2.1**				
37″.5...49″.4	*n*	0	0	0	0	0	0	0	0
	N								
49″.5...65″.4	*n*	0	1.0	0	0	0	0	0	0
	N		**1.0**						
$\geqslant$65″.5	*n*	0	0	0	0	0	0	0	0
	N								

Table 13.

U. O. C. 47 (1 Pair of Paris Plates).

Examined once by Innes. Interval 26.9 years.

$p = \pi(s).\ \chi(m).$

$s = 1''.25\ \ 1''.78\ \geqslant 2''.30$

assumed $\pi = 0.79\ \ 0.97\ \ \ 1.00.$

$\chi(m)$ given in table 4.

Probable error in $s = \pm 0''.24$; distribution of proper motions not corrected for error-dispersion. Distribution of magnitudes corrected for error-dispersion with *p. e.* $= \pm 0.38$ st. mg.

The region falls within the galactic zone $0^0...20^0$. Area $= 4\square^0$.

Centennial *p. m.*		Photographic magnitude, *H. S.*									
		$\leqslant 6.44$	7	8	9	10	11	12	13	14	15
4″.0...5″.4	*n*	0	0	1.0	0	1.0	1.4	1.6	4.1	3.3	4.6
	N			**1.3**		**1.3**	**1.9**	**2.2**	**6.4**	**5.0**	**15.2**
5″.5...7″.4	*n*	0	0	0	0	1.0	3.0	4.5	7.0	5.3	4.2
	N					**1.2**	**3.5**	**5.0**	**8.4**	**6.2**	**11.4**
7″.5...9″.4	*n*	0	0	0	0	0	3.0	3.1	3.0	2.2	1.6
	N						**3.2**	**3.5**	**3.4**	**2.5**	**4.0**
9″.5...12″.4	*n*	0	0	0	0	0	1.0	0.9	0.6	0.3	0.1
	N						**1.2**	**1.0**	**0.6**	**0.3**	**0.3**
12″.5...16″.4	*n*	0	0	0	0	0	0.6	0.5	0.6	0.2	0
	N						**0.7**	**0.6**	**0.6**	**0.2**	
16″.5...21″.4	*n*	1.0	0	0	0	0	0	0	0.3	0.3	0.4
	N	**1.1**							**0.4**	**0.3**	**1.1**
21″.5...28″.4	*n*	0	0	0	0	0	0.4	0.2	0.3	0	0
	N						**0.5**	**0.3**	**0.3**		
28″.5...37″.4	*n*	0	0	0	0	0	0.2	0.3	0.3	0.2	0
	N						**0.2**	**0.3**	**0.3**	**0.2**	
37″.5...65″.4	*n*	0	0	0	0	0	0	0	0	0	0
	N										
65″.5...86″.4	*n*	0	0	0	0	1.0	0	0	0	0	0
	N					**1.1**					
$\geqslant$86″.5	*n*	0	0	0	0	0	0	0	0	0	0
	N										

Table 14.

U. O. C. 48 (Sydney Plates).

Searched once by Innes; partly overlapping.
Average interval = 13.4 years.

$$p = \pi(s) \cdot \chi(m).$$

	$s =$	1″.0	1″.2	1″.4	1″.6	1″.8	2″.0	2″.4	2″.8	3″.2	3″.6	4″.0	5″.0	≥6″.0
assumed	$\pi =$	0.10	0.18	0.27	0.34	0.43	0.51	0.64	0.73	0.80	0.85	0.89	0.96	1.00.

$\chi(m)$ given in table 4.

Corrected for error-dispersion, assuming *p. e.* in $s = \pm 0''.60$, and *p. e.* in m as follows:

magn., *H. S.*	≤ 9	10	11	12	13, 14
p. e.	±0.18	±0.24	±0.28	±0.35	±0.38

Centennial *p. m.*		Photographic magnitude, *H. S.* ≤6.44	7	8	9	10	11	12	13
		a) Gal. lat. 0°...20°							
21″.5...28″.4	*n*	0.3	0.6	4.7	5.9	8.2	4.1	0.6	2.0
	N	**0.5**	**0.8**	**7.5**	**7.8**	**12.4**	**7.0**	**1.1**	**4.3**
area, □°		420	420	420	420	420	420	390	330
		b) Gal. lat. 20°...40°							
21″.5...28″.4	*n*	0.6	0.7	0.9	0.9	0.8	0.7	0.8	0.3
	N	**0.8**	**1.1**	**1.2**	**1.8**	**1.5**	**1.1**	**1.4**	**0.4**
area, □°		100	100	100	100	100	100	95	80
		c) Gal. lat. 40°...90°							
21″.5...28″.4	*n*	0	0	2.4	2.1	4.4	2.5	1.5	0.8
	N			**4.1**	**3.2**	**6.8**	**4.1**	**2.5**	**1.5**
area, □°		180	180	180	180	180	180	165	140
		d) Gal. lat. 0°...90°							
area, □°		700	700	700	700	700	700	650	550
28″.5...37″.4	*n*	0.7	0.9	7.8	9.2	8.3	6.4	4.5	2.3
	N	**1.2**	**1.1**	**10.4**	**13.4**	**11.2**	**9.2**	**6.7**	**4.4**
37″.5...49″.4	*n*	0.1	1.1	3.7	4.0	3.5	3.1	2.9	1.3
	N	**0.2**	**1.2**	**4.5**	**4.8**	**4.1**	**3.7**	**3.6**	**2.0**
49″.5...65″.4	*n*	1.0	0.3	1.8	1.8	4.2	2.4	1.7	2.0
	N	**1.1**	**0.3**	**2.0**	**2.0**	**4.7**	**2.7**	**2.0**	**2.9**
65″.5...86″.4	*n*	1.0	0	0.2	0.4	1.4	1.4	2.5	1.1
	N	**1.1**		**0.2**	**0.4**	**1.6**	**1.4**	**3.0**	**1.3**
86″.5...110″.4	*n*	1.0	0	2.0	0	2.2	0.8	0	0
	N	**1.1**		**2.2**		**2.4**	**0.9**		
110″.5...146″.4	*n*	0	0	0	0	1.0	0.2	1.3	1.5
	N					**1.1**	**0.2**	**1.5**	**2.3**
146″.5...194″.4	*n*	0	0	1.0	0	0.2	0.8	0	0
	N			**1.1**		**0.2**	**0.9**		
≥ 194″.5	*n*	0	1.0	0	0	0	0.4	1.6	0
	N		**1.1**				**0.4**	**1.9**	

Table 15.

U. O. C. 49 (Algiers Plates).

The chart plates were not used, as the interval is too short.

Examined once by Innes. Average interval = 24 years.

$$p = \pi(s).\,\chi(m).$$

$s =$	0″.7	0″.8	0″.9	1″.0	1″.2	1″.4	1″.6	1″.8	2″.0	2″.4	2″.8	3″.2	3″.6	⩾4″.0
assumed $\pi =$ (indirectly determined)	0.14	0.26	0.39	0.50	0.64	0.74	0.81	0.87	0.90	0.93	0.95	0.97	0.99	1.00

$\chi(m)$ given in table 4.

Proper motions were not corrected for error-dispersion.

Distribution of magnitudes was corrected for error-dispersion, assuming the *p. e.* = ± 0.35 st. mg.

a) Gal. lat. 20°....40°; area = 14□° to 11 *H. S.*, and 5□° for 12 *H. S.*

Centennial *p. m.*		Photogr. magn., *H. S.* ⩽7.44	8	9	10	11	12
7″.5...9″.4	*n*	0	0.2	1.2	0.6	6.4	6.6
	N		**0.2**	**1.4**	**0.8**	**8.6**	**11.1**
9″.5...12″.4	*n*	0	0	0.4	1.1	1.8	0.7
	N			**0.4**	**1.2**	**2.2**	**0.9**
12″.5...16″.4	*n*	0	0.4	2.3	1.8	1.1	0.4
	N		**0.4**	**2.6**	**1.9**	**1.4**	**0.4**
16″.5...21″.4	*n*	0	0.4	1.6	0.5	1.8	0.7
	N		**0.4**	**1.8**	**0.5**	**2.1**	**0.9**
21″.5...28″.4	*n*	0	0	0.4	0.6	0	0
	N			**0.4**	**0.7**		

b) Gal. lat. 40°...90°; area = 10□° to 11 *H. S.*, and 5□° for 12 *H. S.*

Centennial *p. m.*		Photogr. magn., *H. S.* ⩽7.44	8	9	10	11	12
7″.5...9″.4	*n*	0	0	0.7	2.2	1.1	0
	N			**0.9**	**2.9**	**1.4**	
9″.5...12″.4	*n*	0	0.2	1.5	3.2	2.1	0
	N		**0.2**	**1.8**	**3.6**	**2.4**	
12″.5...16″.4	*n*	0	1.0	0	1.4	2.2	0.4
	N		**1.1**		**1.6**	**2.5**	**0.4**
16″.5...21″.4	*n*	0	0.2	1.5	1.3	0	0
	N		**0.2**	**1.7**	**1.4**		
21″.5...28″.4	*n*	0	0	0.4	1.1	0.5	0
	N			**0.4**	**1.2**	**0.6**	

c) Gal. lat. 20°...90°; area = 24□° to 11 *H. S.*, and 10□° for 12 *H. S.*

Centennial *p. m.*		⩽7.44	8	9	10	11	12
28″.5...37″.4	*n*	0	0	0	0.5	0.5	0
	N				**0.5**	**0.6**	
37″.5...49″.4	*n*	0	1.0	0	0	0	0
	N		**1.1**				
⩾49″.5	*n*	0	0	0	0	0	0
	N						

Table 16.

U. O. C. 53 (Melbourne Plates)

Searched once by Innes, but about $^5/_{18}$ of the zone is covered twice; some of the data of *U. O. C.* 37 are also included. Average interval = 20 years.

Effective coefficient of perception $p = \pi(s) \cdot \chi(m)$.

$s =$	0″.8	0″.9	1″.0	1″.2	1″.4	1″.6	1″.8	2″.0	2″.4	2″.8	3″.2	3″.6	⩾4″.0
not overlapping, $\pi =$	0.12	0.20	0.29	0.47	0.61	0.69	0.76	0.81	0.86	0.91	0.94	0.95	0.96
1) overlapping, effective $\pi =$	0.46	0.52	0.59	0.69	0.77	0.84	0.91	0.96	1.04	1.07	1.09	1.10	1.10
assumed effective π whole zone	0.21	0.29	0.37	0.53	0.66	0.73	0.80	0.85	0.91	0.95	0.98	0.99	1.00

$\chi(m)$ given in table 4.

Corrected for error-dispersion, assuming:

p. e. in $s = \pm 0''.48$ (not overlapping) and $\pm 0''.34$ (overlapping regions);

p. e. in *m*: magn., *H. S.*	⩽ 10	11	12	⩾ 13
p. e.	±0.18	±0.26	±0.34	±0.38

a) Gal. lat. 0°...20°; area = 53□°

Centennial *p.m.*		Photogr. magn., *H. S.* ⩽6.44	7	8	9	10	11	12	13
7″.5...9″.4	*n*	0.3	0.3	0.8	5.5	14.1	6.4	8.1	12.3
	N	**0.4**	**0.5**	**1.7**	**8.6**	**23.3**	**11.4**	**13.5**	**19.2**
9″.5...12″.4	*n*	0.7	0.1	0.5	4.1	13.5	5.2	6.4	9.5
	N	**0.8**	**0.2**	**0.9**	**5.4**	**17.5**	**7.3**	**8.6**	**14.0**
12″.5...16″.4	*n*	0.8	0	0.6	1.7	4.8	2.0	2.3	2.7
	N	**1.0**		**0.7**	**1.9**	**5.2**	**2.4**	**2.8**	**3.3**
16″.5...21″.4	*n*	0.2	0	0.5	1.7	2.0	0.3	1.0	2.1
	N	**0.2**		**0.5**	**1.9**	**2.2**	**0.3**	**1.2**	**2.4**
21″.5...28″.4	*n*	0	0	0.1	1.0	0.9	0	0.7	1.8
	N			**0.1**	**1.1**	**1.0**		**0.8**	**2.1**

b) Gal. lat. 20°...40°; area = 79□° to 12 *H. S.*, and 76□° for 13 *H. S.*

Centennial *p.m.*		Photogr. magn., *H. S.* ⩽6.44	7	8	9	10	11	12	13
7″.5...9″.4	*n*	0	0.8	2.3	3.9	8.5	9.8	16.9	9.4
	N		**1.2**	**4.3**	**7.2**	**13.5**	**15.4**	**27.9**	**17.6**
9″.5...12″.4	*n*	0.3	1.8	1.9	4.6	10.7	13.0	20.9	10.7
	N	**0.4**	**2.3**	**2.8**	**6.2**	**13.9**	**17.3**	**27.6**	**15.0**
12″.5...16″.4	*n*	0.6	2.0	1.2	3.1	5.5	5.7	10.4	6.2
	N	**0.7**	**2.3**	**1.5**	**3.3**	**6.6**	**6.6**	**12.6**	**7.8**
16″.5...21″.4	*n*	0.2	0.7	0.7	1.3	3.1	1.9	2.6	3.0
	N	**0.2**	**0.8**	**0.8**	**1.4**	**3.4**	**2.2**	**2.7**	**3.3**
21″.5...28″.4	*n*	0	0	0	0.2	1.8	1.8	2.7	0.9
	N				**0.2**	**2.0**	**2.1**	**3.0**	**1.0**

1) Examined twice.

Table 16. Continued.

Centennial *p.m.*		Photogr. magn., *H. S.* ≤6.44	7	8	9	10	11	12	13
		c) Gal. lat. $40^0...90^0$; area $= 91\square^0$ to 11 *H. S.*, $87\square^0$ for 12 *H. S.* and $82\square^0$ for 13 *H. S.*							
7″.5...9″.4	*n*	0	1.0	4.4	4.4	9.3	12.6	13.6	7.5
	N		**1.5**	**6.9**	**6.9**	**16.6**	**20.4**	**21.8**	**12.2**
9″.5...12″.4	*n*	0	0.8	4.6	5.7	9.0	13.6	19.1	9.7
	N		**1.1**	**6.4**	**7.3**	**12.3**	**18.0**	**25.8**	**13.3**
12″.5...16″.4	*n*	0	0.1	1.8	3.2	3.1	6.7	8.6	5.7
	N		**0.1**	**2.1**	**3.7**	**3.8**	**7.9**	**10.3**	**7.2**
16″.5...21″.4	*n*	0	0	0.9	1.0	0.2	2.7	5.8	2.2
	N			**1.0**	**1.1**	**0.2**	**3.1**	**6.5**	**2.5**
21″.5...28″.4	*n*	0	0	1.1	1.3	2.7	4.9	2.4	2.3
	N			**1.2**	**1.3**	**2.9**	**5.3**	**2.8**	**2.6**

Centennial *p.m.*		Photogr. magn., *H. S.* ≤6.44	7	8	9	10	11	12	13
		d) Gal. lat. $0^0...90^0$; area $= 223\square^0$ to 11 *H. S.*, $219\square^0$ for 12 *H. S.* and $211\square^0$ for 13 *H. S.*							
28″.5...37″.4	*n*	1.0	1.0	2.3	5.0	3.6	4.4	0.7	1.2
	N	**1.0**	**1.1**	**2.5**	**5.5**	**3.9**	**4.8**	**0.8**	**1.4**
37″.5...49″.4	*n*	0	0	1.0	1.2	1.8	0.3	0.7	1.0
	N			**1.1**	**1.2**	**2.0**	**0.3**	**0.8**	**1.0**
49″.5...65″.4	*n*	0	0	0	0	0	0	0	1.0
	N								**1.2**
65″.5...86″.4	*n*	1.0	1.0	0	0	0	0.3	2.1	0.6
	N	**1.1**	**1.1**				**0.4**	**2.2**	**0.6**
86″.5...110″.4	*n*	0	0	0	0	0	0	0	1.0
	N								**1.2**
110″.5...146″.4	*n*	0	0	0	0	0	0	0	0
	N								
146″.5...194″.4	*n*	1.0	0	0	0.1	0.9	0	0	1.0
	N	**1.0**			**0.1**	**0.9**			**1.2**
≥194″.5	*n*	1.0	0	0	0	0	0	0	0
	N	**1.1**							

Table 17.

U. O. C. 54 (Cape South Polar Plates).

Searched by Innes; the overlaps are considerable. Interval = 21.4 years. For a region covered ν times the coefficient of perception was assumed equal to

$$p_\nu = 1-[1-\pi(s) \, . \, \chi(m)]^\nu,$$

where $\chi(m)$ is given by table 4, and $\pi(s)$ was taken from table 5. As both, the magnitudes and the proper motions, are determined relatively accurately, no correction for error-dispersion was made.

The whole region falls within galactic latitude 20°...40°.

The area is 13 □°.

Centennial *p. m.*		Photogr. magn., *H. S.*						
		⩽ 6.44	7	8	9	10	11	12
5″.5...7″.4	*n*	0	0	1.0	0	2.0	4.0	2.0
	N			**1.8**		**3.6**	**10.9**	**10.7**
7″.5...9″.4	*n*	0	1.0	0	1.0	0	1.0	3.0
	N		**1.5**		**1.6**		**1.1**	**7.6**
9″.5...12″.4	*n*	0	0	0	1.0	1.0	2.0	2.0
	N				**1.0**	**1.3**	**2.6**	**3.8**
12″.5...16″.4	*n*	0	0	1.0	1.0	1.0	2.0	0
	N			**1.0**	**1.2**	**1.0**	**2.0**	
16″.5...21″.4	*n*	0	0	0	2.0	0	0	0
	N				**2.1**			
21″.5...28″.4	*n*	0	0	0	0	1.0	0	0
	N					**1.0**		
28″.5...37″.4	*n*	0	0	0	0	0	0	0
	N							
37″.5...49.″4	*n*	0	0	0	0	0	1.0	0
	N						**1.0**	
⩾ 49″.5	*n*	0	0	0	0	0	0	0
	N							

Table 18.

Hels. I.

Examined by Furuhjelm. Average interval 18.3 years.

Proper motions exceeding 49″.5 are included in table 20.

Assumed values of the coefficient of perception, p, and the extrapolation factor, $\bar{z}$ = mean value of $\frac{1}{p}$:

limits of proper motion:		4″.0...5″.4	5″.5...7″.4	7″.5...9″.4	9″.5...12″.4	⩾12″.5
a) Regions examined once	$\bar{p}$	0.290	0.535	0.720	0.915	0.980
	$\bar{z}$	3.90	1.91	1.40	1.10	1.02
b) Regions examined twice	$\bar{p}$	0.260	0.576	0.830	0.985	1.000
	$\bar{z}$	4.35	1.77	1.21	1.03	1.00
c) Regions examined 3 or more times	$\bar{p}$	0.418	0.740	0.935	1.000	1.000
	$\bar{z}$	2.70	1.38	1.08	1.01	1.00

The values of p for μ = 4″.0...5″.4 are slightly extrapolated.

The coefficient of perception does not depend upon the magnitude except for *H. S.* 12; in this case some of the stars of the Helsingfors Astrographic catalogue could not be observed, but as a compensation may be regarded a number of faint stars observed by Furuhjelm which were not included in the Astrographic catalogue. The effective limiting magnitude of the Astrographic catalogue is 12.49 *H. S.* (table IV of *G. P.* **27**); for the stars observed by Furuhjelm the limiting magnitude was assumed at 12.44 *H. S.*

Both the proper motions and magnitudes being given with high accuracy, no correction for error-dispersion was introduced.

The whole region falls within the limits of galactic latitude 40°...90°.

The area equals 268 □°.

Centennial p. m.		Photogr. magn., *H. S.* ⩽6.44	7	8	9	10	11	12
4″.0...5″.4	n	2	2	3	7	36	37	48
	N	**7.0**	**7.0**	**13.2**	**23.9**	**133.4**	**136.8**	**187.7**
5″.5...7″.4	n	3	2	8	16	42	57	52
	N	**5.0**	**3.6**	**13.2**	**26.6**	**68.8**	**95.1**	**92.5**
7″.5...9″.4	n	0	4	8	12	31	36	47
	N		**4.4**	**9.5**	**14.5**	**37.4**	**44.0**	**59.3**
9″.5...12″.4	n	2	4	12	11	41	25	41
	N	**2.1**	**4.1**	**12.6**	**11.5**	**43.1**	**26.0**	**43.5**
12″.5...16″.4	n	4	4	6	10	21	19	23
	N	**4.0**	**4.0**	**6.0**	**10.0**	**21.1**	**19.1**	**23.1**
16″.5...21″.4	n	3	1	2	2	21	13	9
	N	**3.0**	**1.0**	**2.0**	**2.0**	**21.2**	**13.1**	**9.1**
21″.5...28″.4	n	1	1	2	3	8	9	10
	N	**1.0**	**1.0**	**2.0**	**3.0**	**8.1**	**9.0**	**10.1**
28″.5...37″.4	n	2	1	1	1	1	1	4
	N	**2.0**	**1.0**	**1.0**	**1.0**	**1.0**	**1.0**	**4.1**
37″.5...49″.4	n	0	0	0	1	0	1	2
	N				**1.0**		**1.0**	**2.0**

Table 19.

U. O. C. 58.

List of proper motions exceeding 0″.50 annually found in *C. A. Z.* by Innes and Wood.

Corrected for error-dispersion in magnitudes only, which correction introduced little change: it appeared sufficient to join estimated magn. 13 and 14 and assume the sum to represent 13 *H. S.* Area = 1500 □°.

Centennial *p. m.*		Photogr. magn., *H. S.*							
		⩽6.44	7	8	9	10	11	12	13
49″.5...65″.4	*n* *N*	1 **1.0**	1 **1.0**	5 **5.2**	1 **1.0**	2 **2.1**	4 **4.1**	4 **4.1**	3 **3.1**
65″.5...86″.4	*n* *N*	1 **1.0**	0	2 **2.0**	2 **2.0**	2 **2.0**	2 **2.0**	4 **4.0**	2 **2.2**
86″.5...110″.4	*n* *N*	0	2 **2.0**	1 **1.0**	0	1 **1.0**	2 **2.0**	0	2 **2.2**
110″.5...146″.4	*n* *N*	0	0	0	0	1 **1.0**	0	1 **1.0**	1 **1.2**
146″.5...194″.4	*n* *N*	0	1 **1.0**	0	0	0	1 **1.0**	1 **1.0**	0
⩾ 194″.5	*n* *N*	1 **1.0**	0	0	0	0	1 **1.0**	0	0

Table 20.

Hels. a.

Coefficients of perception are estimated.

Area = 2060 □°.

Centennial *p. m.*		Photogr. magn., *H. S.*						
		⩽6.44	7	8	9	10	11	12
49″.5...65″.4	*n* *N*	1 **1.0**	1 **1.0**	2 **2.0**	2 **2.1**	4 **4.4**	3 **3.4**	8 **11.9**
65″.5...86″.4	*n* *N*	2 **2.0**	2 **2.0**	4 **4.1**	4 **4.2**	4 **4.4**	4 **4.6**	3 **4.5**
86″.5...110″.4	*n* *N*	0	0	0	0	0	1 **1.1**	1 **1.5**
110″.5...146″.4	*n* *N*	0	0	0	1 **1.0**	0	0	0
146″.5...194″.4	*n* *N*	0	0	0	0	1 **1.1**	0	0
⩾ 194″.5	*n* *N*	0	0	0	1 **1.0**	1 **1.1**	0	0

search is expressed in square degrees; it is the real area, or the total area covered by the plates, *minus* the probable loss due to the *réseau*, the overlaps by neighbouring plates, etc. For the faintest magnitudes the area is frequently assumed to be smaller than for the brighter stars, because occasionally some pairs of plates did not reach the average limiting magnitude of the series.

The separate results are contained in tables 9—20. Components of double stars or pairs having common proper motion were counted as one star.

4. Final Results.

Tables 21 and 23 contain the final result. Table 21 gives the concluded true number of stars within given limits of proper motion, photographic magnitude and galactic latitude, separately for the observations of Innes (*U. O. C.*) and of Furuhjelm (*Hels*).

The true number was computed from the data of tables 9—20, according to the formula

$$\text{true number per } 10000\,\square^0 = \Sigma N \times \frac{10000}{\text{Area}} \ldots \text{(a)}.$$

This formula is to be preferred in the case when real variations in the density of proper motion stars in different regions of the sky are greater than produced by a purely chance distribution. Were it otherwise, i. e. were the proper motions distributed according to the law of chance, the following formula should be used:

$$\text{number per } 10000\,\square^0 = \Sigma n \times \frac{10000}{\text{Effective Area}} \ldots \text{(b)},$$

the effective area being defined by equation (5), p. 6.

The agreement of the different series represented in table 21 is generally as good as might be expected from the character of these data; the differences between *Hels.* and *U. O. C.* are in most cases not greater than the differences between the hree zones of *U. O. C.* themselves; these differences may be attributed partly to systematic errors in the magnitude scale or in the coefficient of perception; the effect of galactic latitude appears to be small even for the smallest proper motions. Table 22 makes the comparison of the different series easier as

Table 21.

True number of proper motion stars per 10000□°.

Centen. *p.m.*, Source and Gal. Lat. \ Photogr. Magn.		≤6.44	7	8	9	10	11	12	13	≤11	12	13
		Number per 10000□°								Area □°		
4″.0...5″.4	*U.O.C.* 0°—20°	450	0	760	0	2200	2500	5100	9700	33	33	33
	Hels. 40—90	260	260	490	890	4970	5100	7000	—	268	268	—
5″.5...7″.4	*U.O.C.* 0—20	0	0	390	670	2900	2600	3900	7000	33	33	33
	„ 20—40	0	0	1400	0	2800	8400	8200	—	13	13	—
	Hels. 40—90	190	130	490	990	2570	3550	3450	—	268	268	—
7″.5...9″.4	*U.O.C.* 0—20	50	60	200	1130	3620	2460	2620	3610	86	86	86
	„ 20—40	0	250	420	960	1340	2370	4800	2320	106	97	76
	„ 40—90	0	150	680	770	1930	2160	2370	1490	101	92	82
	Hels. 40—90	0	160	350	540	1390	1640	2210	—	268	268	—
9″.5...12″.4	*U.O.C.* 0—20	70	120	290	710	1400	1970	1020	1650	324	230	225
	„ 20—40	20	160	300	590	1130	1890	2450	1870	209	165	143
	„ 40—90	20	120	470	980	1500	2080	1770	1580	243	189	177
	Hels. 40—90	80	150	470	430	1610	970	1620	—	268	268	—
12″.5...16″.4	*U.O.C.* 0—20	60	140	170	650	820	1280	490	830	355	230	225
	„ 20—40	40	180	330	380	640	850	1110	920	223	165	143
	„ 40—90	0	60	240	540	870	1230	940	950	262	189	177
	Hels. 40—90	150	150	220	370	790	710	860	—	268	268	—
16″.5...21″.4	*U.O.C.* 0—20	40	120	100	300	360	400	320	420	355	230	225
	„ 20—40	70	80	130	280	240	450	360	430	223	165	143
	„ 40—90	0	20	80	290	290	570	520	410	262	189	177
	Hels. 40—90	110	40	70	70	790	490	340	—	268	268	—
21″.5...28″.4	*U.O.C.* 0—20	6	28	98	168	245	253	43	156	775	620	555
	„ 20—40	50	70	40	110	190	230	180	70	323	260	223
	„ 40—90	0	0	124	210	330	372	180	170	442	354	317
	Hels. 40—90	40	40	70	110	300	340	380	—	268	268	—
28″.5...37″.4	*U.O.C.* 0—90	14	23	126	163	130	145	77	104	1540	1234	1095
	Hels. 40—90	70	40	40	40	40	40	150	—	268	268	—
37″.5...49″.4	*U.O.C.* 0—90	1	23	65	60	55	66	49	50	1540	1234	1095
	Hels. 40—90	0	0	0	40	0	40	70	—	268	268	—
49″.5...65″.4	*U.O.C.* 0—90	11	8	29	14	24	24	23	32	2866	2734	2595
	Hels. 0—90	5	5	10	10	21	16	58	—	2060	2060	—
65″.5...86″.4	*U.O.C.* „	11	4	12	8	16	13	34	16	2866	2734	2595
	Hels. „	10	10	20	20	21	22	22	—	2060	2060	—
86″.5...110″.4	*U.O.C.* „	4	7	11	0	12	10	0	12	2866	2734	2595
	Hels. „	0	0	0	0	0	5	8	—	2060	2060	—
110″.5...146″.4	*U.O.C.* „	0	0	0	0	7	1	9	13	2866	2734	2595
	Hels. „	0	0	0	5	0	0	0	—	2060	2060	—
146″.5...194″.4	*U.O.C.* „	7	3	4	0	8	7	6	6	2866	2734	2595
	Hels. „	0	0	0	0	5	0	0	—	2060	2060	—
≥194″.5	*U.O.C.* „	7	4	0	0	5	8	7	0	2866	2734	2595
	Hels. „	0	0	0	5	5	0	0	—	2060	2060	—

Table 22.

Comparison of the observations of Innes and of Furuhjelm.

True number per 10000□° of stars brighter than photographic magnitude 12.45.

Centennial p. m.	4″.0... ...5″.4	5″.5... ...7″.4	7″.5... ...9″.4	9″.5... ...12″.4	12″.5... ...16″.4	16″.5... ...21″.4	21″.5... ...28″.4
I., 0°—20° Gal. Lat.	11000	10460	10140	5580	3610	1640	841
I., 20—40 „ „	—	20800	10140	6540	3530	1610	870
I., 40—90 „ „	—	—	8060	6940	3880	1770	1216
F., 40—90 „ „	18970	11370	6290	5330	3250	1910	1280

Centennial p. m.	28″.5... ...37″.4	37″.5... ...49″.4	49″.5... ...65″.4	65″.5... ...86″.4	86″.5... ...110″.4	110″.5... ...146″.4	146″.5... ...194″.4	⩾194″.5
I., 0°—90° Gal. Lat.	678	319	133	98	44	17	35	29
F., 40°—90° „ „	420	150	125	125	13	5	5	10

the effect of magnitude is there practically eliminated. Between the *U. O. C.* and the *Hels.* series there may be noted, however, a striking discordance for which cannot be made responsible any of the causes mentioned above; the discordance takes place for the largest proper motions, where it could be expected the least. The number of proper motions exceeding 0″.865 annually and brighter than $m = 12.45$ as derived from the *U. O. C.*, results about 4 times greater than the number furnished by *Hels.*; the observed numbers are 34 per 2866□° in *U. O. C.* and 6 per 2060□° in *Hels.*; a chance difference of such a size is very improbable; it appears that the largest proper motions show a real concentration in the southern hemisphere; this is confirmed by other sources; for instance, stars brighter than visual magnitude 7.45, for which Luyten's list in Lick Observatory Bull. **344** may be regarded as complete, show the same phenomenon.

As shown by table 22, certainly no galactic condensation is revealed by stars of a proper motion greater than 0″.095. As

Table 23.

True number of stars per 10000 □°.

Joined results for the Innes and the Furuhjelm stars, without regard to galactic latitude. The observed number is added in parentheses.

Photogr. Magn. / Centennial *p. m.*	$\leqslant$6.44	7	8	9	10	11	12	13	$\geqslant$11	12	13
	Number per 10000 □°								Area □°		
4″.0...5″.4	280 (3.0)	230 (2.0)	520 (5.0)	790 (7.0)	4690 (41.0)	4820 (43.4)	6800 (59.6)	9700 (22.5)	301	301	33
5″.5...7″.4	160 (3.0)	110 (2.0)	520 (10.0)	920 (18.0)	2610 (52.0)	3640 (68.0)	3700 (65.5)	7000 (19.4)	314	314	33
7″.5....9″.4	10 (0.3)	160 (7.1)	400 (15.7)	750 (29.7)	1820 (72.7)	2000 (82.3)	2770 (103.3)	2480 (39.8)	561	543	244
9″.5...12″.4	52 (4.3)	137 (10.3)	381 (28.3)	674 (46.9)	1420 (109.4)	1739 (117.3)	1650 (107.7)	1690 (57.7)	1044	852	545
12″.5...16″.4	64 (6.8)	131 (12.5)	233 (21.9)	502 (44.1)	787 (71.2)	1046 (89.4)	826 (59.7)	900 (35.3)	1108	852	545
16″.5...21″.4	52 (5.6)	70 (7.0)	94 (9.6)	238 (23.6)	425 (43.4)	474 (45.5)	371 (29.1)	411 (18.6)	1108	852	545
21″.5...28″.4	17 (2.6)	30 (4.5)	90 (11.4)	160 (24.1)	263 (38.4)	290 (40.6)	160 (20.7)	153 (12.4)	1808	1502	1095
28″.5...37″.4	23 (3.7)	25 (4.2)	113 (17.0)	145 (21.0)	116 (17.5)	129 (18.5)	91 (11.0)	104 (8.0)	1808	1502	1095
37″.5...49″.4	1 (0.1)	19 (3.2)	55 (8.8)	57 (9.1)	47 (7.4)	62 (9.3)	53 (7.0)	50 (4.6)	1808	1502	1095
49″.5...65″.4	8 (4.0)	7 (3.3)	21 (9.8)	12 (5.8)	23 (10.2)	21 (9.4)	38 (13.8)	32 (6.8)	4926	4794	2595
65″.5...86″.4	11 (5.0)	6 (3.0)	15 (7.2)	13 (6.4)	17 (8.4)	16 (7.7)	29 (11.6)	16 (3.7)	4926	4794	2595
86″.5...110″.4	2 (1.0)	4 (2.0)	6 (3.0)	0 (0.0)	7 (3.2)	8 (3.8)	3 (1.0)	12 (3.0)	4926	4794	2595
110″.5...146″.4	0 (0.0)	0 (0.0)	0 (0.0)	2 (1.0)	4 (2.0)	0 (0.2)	5 (2.3)	13 (2.5)	4926	4794	2595
146″.5...194″.4	4 (2.0)	2 (1.0)	2 (1.0)	0 (0.1)	7 (3.1)	4 (1.9)	3 (1.6)	6 (1.3)	4926	4794	2595
$\geqslant$194″.5	4 (2.0)	2 (1.0)	0 (0.0)	2 (1.0)	4 (2.0)	5 (2.2)	4 (1.6)	0 (0.0)	4926	4794	2595

far as the evidence of the table goes, the galactic condensation of proper motions from 0″.040 to 0″.095 must also be small or negligible; however, the data for galactic zones 0°...20° and 20°...40° and proper motions below 0″.075 are too scarce.

From *G. P.* **30** the following data for the different galactic zones may be found:

True number of stars brighter than 12.45 magn. vis., per 10000 □°.

Gal. lat.	Proper motion		
	0″.020 - 0″.040	0″.040—0″.060	0″.060—0″.080
0°...20°	133 000	47 000	16 300
20°...40°	155 000	32 000	18 700
40°...90°	136 000	39 000	17 100

From these data it appears that the galactic condensation of small proper motions such as 0″.02—0″.04 is, at least, doubtful; in a statistics concerning only proper motions exceeding 0″.04 the effect of galactic latitude may thus, in a first approximation, be neglected. Table 23 gives the final result derived from all series of observations here discussed, without regard to galactic latitude.

Table 24 contains a comparison of the results of the present discussion with the data of *G. P.* **30.** For this purpose the data of our table 23 are reduced to the limits of proper motion used in *G. P.* **30**. Average values for the three galactic zones of *G. P.* **30** are assumed. The magnitude scales are used unchanged; as the colour-index may be assumed to lie between 0.0 and +1.0, the visual magnitude m may be assumed to correspond to an average photographic magnitude between m and $m+1$; according to this the data of table 24 are arranged.

Our data derived from blink-microscope observations by using coefficients of perception agree roughly with the results of *G. P.* **30**, though some systematic differences may be traced; a large part of the systematic differences between our results and those of *G. P.* **30** may be attributed to the difference of the photographic and visual scales of magnitude; perhaps, the circumstance that the blink-mikroscope furnishes relative proper motions may account for some difference in the number of small proper motions.

Table 24.

Comparison of our results (*T.*) with *Groningen Publications* **30**

Centennial *p. m.*		Phot. 7	Vis. 7	Phot. 8	Vis. 8	Phot. 9	Vis. 9	Phot. 10	Vis. 10	Phot. 11	Vis. 11	Phot. 12	Vis. 12	Phot. 13	Vis. 13
		True Number per 10 000 □°													
4″.0— 6″.0	*T.*	260		650		1020		5340		5730		7720		11450	
	G. P. **30**		540		1340		3320		6700		11400		15800		—
6″.0— 8″.0	*T.*	120		490		880		2420		3230		3470		5870	
	G. P. **30**		240		440		1730		3030		4100		7600		—
8″.0—10″.0	*T.*	143		364		672		1580		1790		2360		2140	
	G. P. **30**		159		340		870		1400		1940		3770		—
10″.0—15″.0	*T.*	196		462		875		1672		2102		1887		1972	
	G. P. **30**		220		416		1010		1680		1770		2790		—
15″.0-20″.0	*T.*	98		154		356		593		725		569		626	
	G. P. **30**		68		133		420		530		630		793		—
20″.0—30″.0	*T.*	55		137		255		409		453		286		293	
	G. P. **30**		70		121		259		307		320		470		393
> 30″.	*T.*	59		193		207		206		224		211		216	
	G. P. **30**		71		118		175		191		235		333		421

5. Summary.

The method of coefficients of perception is applied to blink-microscope observations, and the number of stars within different limits of proper motion and photographic magnitude is derived from the data published, up to the end of 1924, by Dr. R. T. A. Innes and by Dr. R. Furuhjelm. Tables 21 and 23 contain the final results.

Appendix.

The Luminosity and Density Laws.

The data of table 23 may be used for determining the luminosity and density functions near the sun. The distribution of proper motions for stars of different magnitudes being given, the functions mentioned above may be found if the relation of mean parallax to proper motion and apparent magnitude, and the error-function of this mean parallax are given. The knowledge of the mean parallax is the most important one, and different assumptions as to this function may lead to results differing as widely as those of Kapteyn and Van Rhijn [1]) and those of Seares [2]). The error-function of the mean parallax has not such a great influence upon the result, and at present Kapteyn's assumption that for a given mean parallax the logarithms of the true parallaxes are distributed according to a Gaussian error-curve with a probable error $= \varrho$ may be regarded as the best first approximation. The unknown effect of absorption of light in space will be neglected.

The mean parallax formula, $\bar{\pi}_{m,\mu}$ may be determined only empirically. The scarcity of the material of directly determined parallaxes has impelled different investigators to make simplifying assumptions regarding the character of the function $\bar{\pi}_{m,\mu}$. Thus Kapteyn writes

$$\log \bar{\pi}_{m,\mu} = a + bm + c \log \mu \quad \ldots\ldots \text{(a); however,}$$

the coefficients actually used by Kapteyn [1]) correspond even to a more particular case

$$\left.\begin{aligned} \overline{M} &= A + BH, \\ H &= m + 5 \log \mu, \\ \overline{\log \pi} &= \frac{\overline{M} - m}{5} \end{aligned}\right\} \text{(b).}$$

1) *Mt. Wilson Contributions* **188.**

2) *Mt. Wilson Contr.* **278.**

Here $\overline{M}$ is the mean absolute magnitude, m — the apparent magnitude, μ — the proper motion. The difference of the logarithmic and arithmetic mean of the parallax must be taken into account in the transition from (b) to (a).

Seares uses two linear relations, instead of one, of the form $M = A + BH$, one for low, the other for high values of H. The second straight line for stars of low luminosity was introduced to satisfy the observed parallaxes of absolutely faint stars, for which Kapteyn's formula, as well as table 19 of *G. P.* **34**, furnishes far too small values.

In *Harvard Circular* **262** Luyten criticizes the method of Seares, pointing at the uncertainty in the coefficient B of the linear formula for the absolutely faint stars. Luyten thinks that the present observational material does not permit of determining the luminosity law with any precision. This conclusion of Luyten must, however, be regarded as somewhat too pessimistic; certainly, the curve of Seares may be regarded as a better approximation to the true curve than Kapteyn's curve, because the expression for $\overline{\pi}_{m,\mu}$ used by Seares represents more or less satisfactorily the parallaxes of absolutely faint stars, whereas Kapteyn's formula utterly fails in this respect.

It appears that the preconceived assumption of a linear relation between M and H accounts for a large part of the difficulty of determining this relation satisfactorily. In trying to represent a curvilinear relation with the aid of a straight line, we shall always meet with a considerable difficulty.

Without losing the advantage of using H as the single argument of M, we may assume generally

$$M = f(H) \quad \ldots\ldots \quad \text{(c)},$$

and from the data furnished by trigonometric parallaxes try to determine the curve $f\ (H)$, imposing the only condition, that of the smoothness of the curve.

In Monthly Notices, **85** (1924), p. 157, Mr. Gorakh Prasad gives a summary of the relation of M to H for different spectral types separately, derived from about 2200 trigonometric parallaxes. His data may be used for determining the relation for all spectral types together. For this purpose the values of $\overline{M} = 5 \log \overline{\pi} - 5 \log (\overline{10^{-0.2m}})$, given by Prasad[1]) for separate spectra, were plotted and a curve drawn which should represent the relation of $\overline{M}$ to H for all spectral types together. In drawing the curve the weight of the individual points was taken into account. The curve with the separate points is shown on fig. 2. Above $H = 11$, or $M = +5.5$ the curve is = an extrapo-

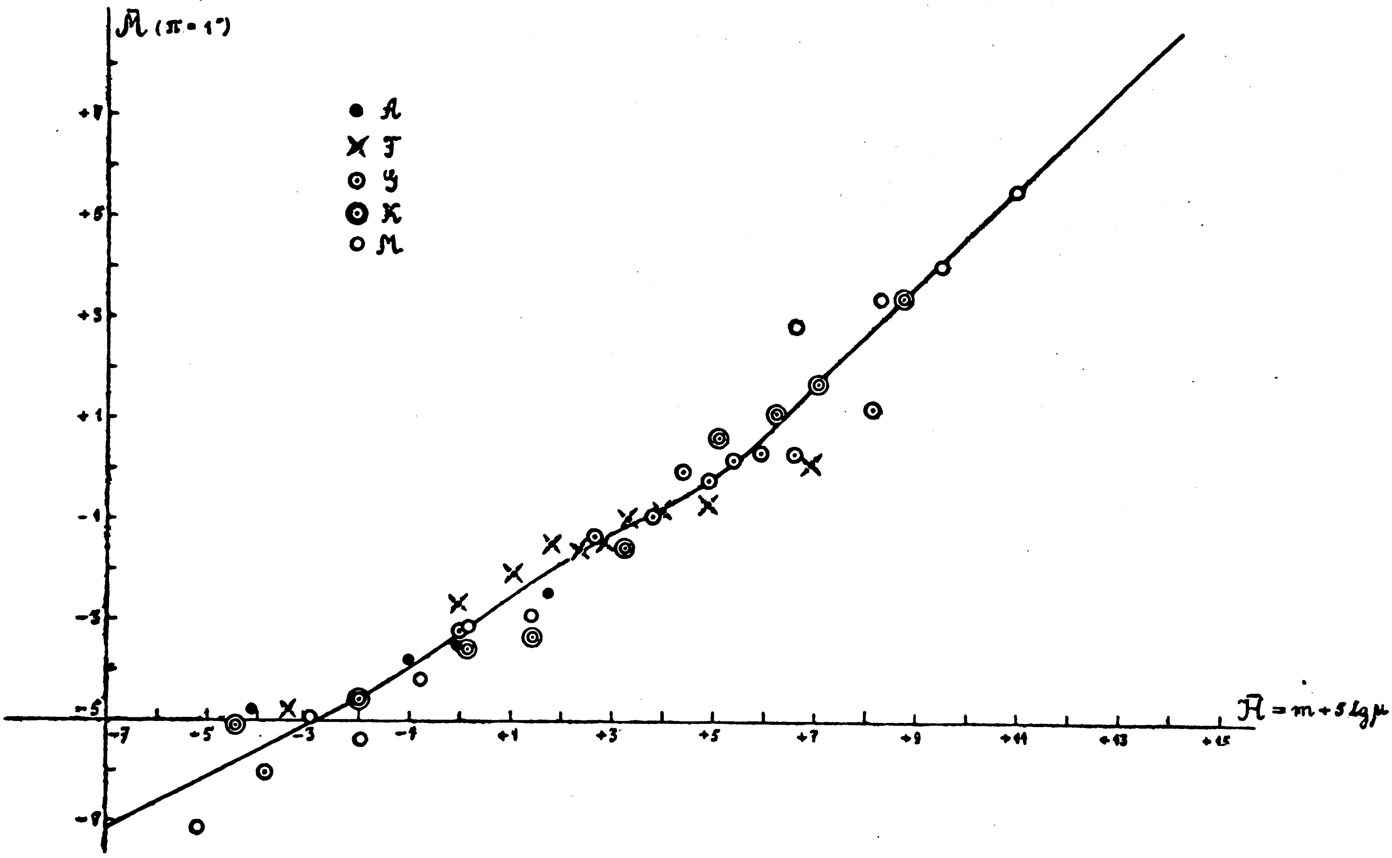

Fig. 2.

lation. Table 25 gives the separate values of $\overline{M}$ read from the curve. It may be remarked that the absolute magnitudes of the table require a correction to reduce them to the arithmetical mean of M, or the geometrical mean of the true parallaxes; the correction may be assumed equal to —0.33 stellar magnitudes, corresponding to a probable error of the true logarithm of π equal to ± 0.16.

1) The absolute magnitudes are here changed to correspond to $\pi = 1''$; the value of $H = m + .5 \log \mu$ is here used, instead of $H = m + 5 + 5 \log \mu$, used by Prasad.

Table 25.

$H = m + 5 \log \mu$, where m is supposed to represent the visual, or the photographic magnitude, according to whether the visual or the photographic absolute magnitudes are to be determined.

H	$\pi=1''$		H	$\pi=1''$		H	$\pi=1''$	
	$\overline{M}_{vis.}$	$\overline{M}_{phtgr.}$		$\overline{M}_{vis.}$	$\overline{M}_{phtgr.}$		$\overline{M}_{vis.}$	$\overline{M}_{phtgr.}$
−7	(−7.12)	(−6.82)	+1	−2.56	−2.45	+9	+3.66	+3.66
−6	−6.62	−6.32	+2	−1.86	−1.74	+10	+4.62	+4.62
−5	−6.12	−5.82	+3	−1.26	−1.10	+11	+5.55	+5.55
−4	−5.57	−5.27	+4	−0.82	−0.54	+12	(+6.50)	(+6.50)
−3	−5.08	−4.78	+5	−0.17	+0.04	+13	(+7.45)	(+7.45)
−2	−4.55	−4.25	+6	+0.60	+0.76	+14	(+8.40)	(+8.40)
−1	−3.97	−3.77	+7	+1.63	+1.63	+15	(+9.35)	(+9.35)
0	−3.28	−3.16	+8	+2.62	+2.62			

The data for the photographic magnitudes were derived from a preliminary knowledge of the average colour of stars of different absolute magnitudes.

For absolute magnitudes fainter than +6 the data of the table are extrapolated by tracing a straight line which may be regarded as the continuation of the extreme portion of the observed curve. It is interesting to notice how well observational data are represented by this extremity of the curve. The following data were found for stars fainter than the 10th magnitude and of a proper motion exceeding 0″.5 [1]). In all, 15 such stars with measured parallaxes could be found; the average data for these 15 stars are as follows:

arithmetical mean parallax, measured . . . . 0″.103
„ „ „ according to table 25, . . . 0″.111
„ „ „ according to *G. P.* **34**, . . . 0″.038
„ „ „ according to Kapteyn's formula, 0″.034
mean absolute magnitude = +7.0

It is evident that *G. P.* **34**, as well as Kapteyn's linear formula, furnishes for faint stars of large proper motion quite unreliable values of the parallax; of the 15 stars only one (*B. D.* + 1°4637) has a measured parallax (0″.004) smaller than

1) The most recent data refer to parallaxes of Wolf 134, 219, 358, determined by Van Maanen.

the parallax given by *G. P.* **34** (0″.028). On the contrary, even the extreme portion of table 25 seems to represent the observations quite satisfactorily.

The luminosity and density laws were computed from the data of table 23 by assuming the probable error in log π, as determined from the proper motion and apparent magnitude, equal to ± 0.16, and by using two different sets of data for $\pi_{m,\mu}$: a) by assuming the mean absolute magnitudes, and the corresponding $\pi_{m,\mu}$ as determined by table 25; b) by assuming $\pi_{m,\mu}$ according to table 19 of *G. P.* **34.** The latter case is here considered only for purposes of comparison, whereas solution a) may be regarded as an approximation to the true solution. The method of computation of the luminosity-curve and the density law is essentially the method devised by Kapteyn. The fact that our data refer only to proper motions exceeding 0″.04 requires, however, a slight modification of the method; the correction for the "cosmical error" in the parallaxes can be rigorously made only when the distribution of all proper motions from zero to ∞ is given; practically, however, if data for the nearer shells are only to be determined, the knowledge of the number of stars of small proper motions need not be very accurate; the corresponding numbers of stars of small proper motion, i. e. of small $\pi_{m,\mu}$ were therefore approximately determined by a process of extrapolation of the data of the table; the extrapolation was made on a preliminary assumption regarding the density law for $\pi < 0''.006$, and the luminosity-law for $M < -3.5$. An error in the extrapolated figures may influence the final results but very slightly, and can have effect only upon the bright extremity of the luminosity-curve, or upon the density of the farthest shells. The extrapolated numbers being used only for the purpose of freeing the remaining portion of the table from the effect of the "cosmical error", their exact values are only of a conventional nature.

Tables 26 and 27 contain the results of both solutions. Stars brighter than magnitude 6.45 were not taken into account. In table 26 the densities computed by Kapteyn and Van Rhijn (Mount Wilson Contributions **188**) are given for comparison.

The luminosity-curve corresponding to solution *a* of table 27 is represented on fig. 3. To make the data of table 27 comparable with the results of Kapteyn, or of Seares, the photographic scale of magnitudes must be changed into the visual scale. As the stars here considered belong to the dwarf series, the following average colour-indices, based on the results of *T. P.* 26_3, can be adopted:

Absolute Magn. Photographic	−4. 0	−3. 0	−2. 0	−1.0	0. 0	+1. 0	2. 0	3. 0	4. 0	5. 0	6. 0	≥7. 0
Average Colour-Index	+0.02	+0.17	+0.30	+0.45	+0.59	+0.72	+0.86	+1.02	+1.18	+1.31	(+1.38)	(+1.40)

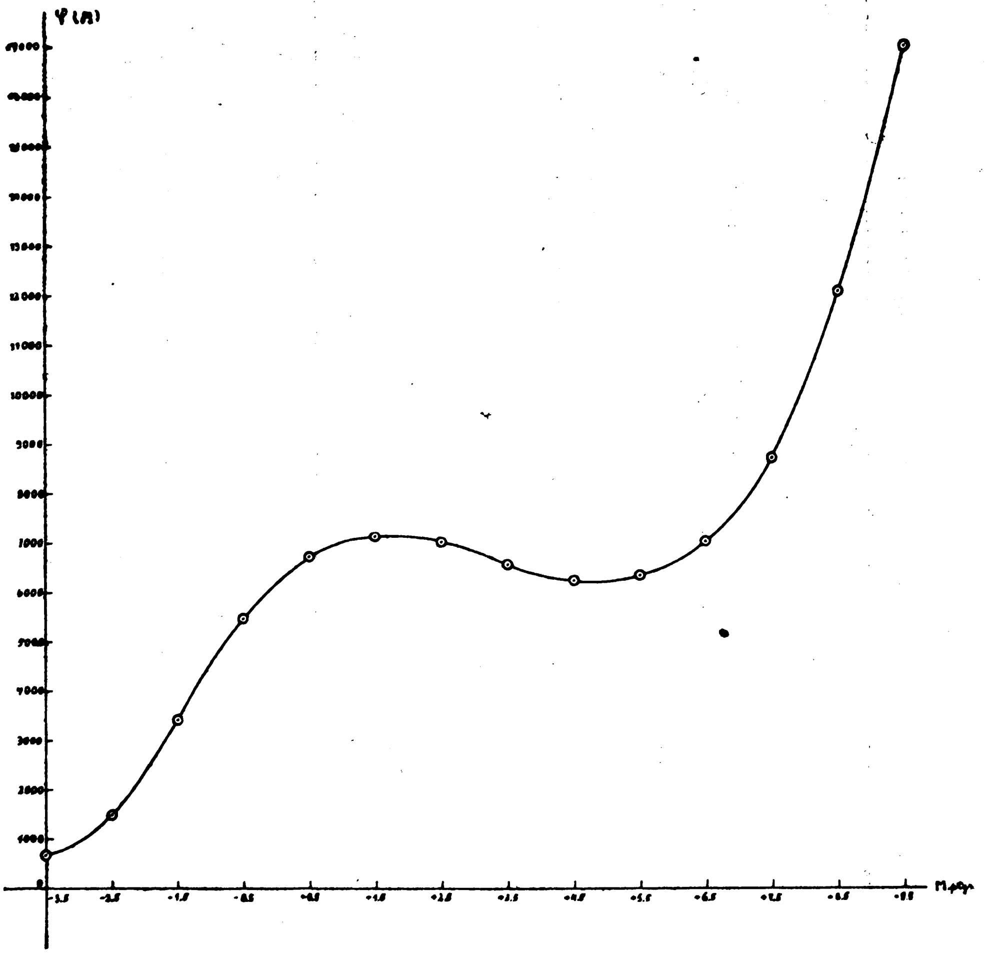

Fig. 3. Frequency function of absolute photographic magnitudes.

Table 26.

Density Law derived from the data of Table 23.

Solution *a* computed with $\pi_{m,\mu}$ as given by table 25 of the present publication.

Solution *b* computed with $\pi_{m,\mu}$ as given by table 19 of *G. P.* **34.**

	Parallax, geom. mean	0″.13	0″.079	0″.050	0″.032	0″.020	0″.013	0″.0079	0″.0050
Density	Solution *a*	1.22	1.14	1.00	0.84	0.67	0.49	0.32	0.20
	Solution *b*	1.18	1.08	1.00	0.89	0.75	0.60	0.47	0.36
	Mt. W. Contr. 188	1.00	1.00	1.00	0.93	0.87	0.77	0.62	0.47

Table 27.

Luminosity-Function of Photographic Magnitudes, derived from the data of table 23.

The meaning of solutions *a* and *b* is explained in table 26.

$M_{ptgr.}$ ($\pi = 1''$)	—3.5	—2.5	—1.5	—0.5	+0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5
	Number of stars per 1 million cubic parsecs of unit density.													
Solution *a*	660	1430	3410	5460	6690	7090	6980	6510	6170	6270	6950	8640	12000	17000
Solution *b*	290	722	1700	3270	5140	6480	6410	5290	3680	2300	1830	1500	—	—
	Effective observed number.													
Solution *a*	6	33	168	324	433	399	269	152	79	41	21.5	11.0	4.5	1.6
Solution *b*	8	62	175	324	444	420	270	135	55	21	7.1	2.1	—	—

Table 28.

Frequency-Function of Visual Absolute Magnitudes.

M		—3.5	—2.5	—1.5	—0.5	+0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5
		Number per 1 million cubic parsecs of unit density.													
From table 23, *T. P.* 26_4	Sol. *a*	920	2350	5070	7140	8090	8290	7620	6990	6900	7690	9980	14000	—	—
	Sol. *b*	419	1190	2820	5040	7070	7600	5920	3670	2260	1720	1300	—	—	—
Kapteyn and V. Rhijn (Mt. W. Contr. 188)		380	910	1830	3200	4790	6100	6860	6430	5290	3960	2410	1510	900	—
Seares (Mt. W. Contr. 273)		550	1270	2410	4050	5700	6970	7400	7200	7700	8800	11700	12900	12900	—

The dispersion of the colour-indices for stars of the dwarf branch is doubtlessly small, and for lack of a precise knowledge we shall neglect this dispersion. Assuming the average colour-indices given above, the data of table 27 were transformed so as to make them correspond to the usually adopted visual scale of absolute magnitudes. The result is given in table 28, together with the results of Kapteyn and Van Rhijn, and of Seares.

In this table solution *a* is to be regarded as representing the most probable true distribution which may be derived from the observational material contained in table 23 of the present publication, because the values of $\pi_{m,\mu}$ are based on trigonometric parallaxes only, without any preconceived assumption regarding the relation of $\overline{M}$ to H except the assumption that $\overline{M}$ depends upon H only.

Solution *b* bears some resemblence to the curve given in *Mt. Wilson Contr.* **188**, the maximum of the curve lying, however, at +1.5, instead of +2.5; the difference is to be attributed 1) to the different expressions for $\pi_{m,\mu}$, the linear formula used by Kapteyn and Van Rhijn being not identical with the data of *G. P.* **34** used for solution *b*; 2) to systematic differences between the data of *G. P.* **30**, which were used by Kapteyn and Van Rhijn, and the distribution of proper motions as found in the present publication; 3) to a difference in the magnitude scales, the magnitude scale used by Kapteyn and Van Rhijn corresponding partly to the true visual scale, partly — for the fainter stars — to a *pseudo-visual* scale, where the systematic variation of colour-index with proper motion has been neglected.

Solution *a* and the luminosity law found by Seares resemble in their general features, though certain systematic differences may be noted, especially concerning the number of bright stars. The difference between both distributions may give an idea of the uncertainty with which the luminosity-curve of stars may at present be calculated, by using quite different observational data and differently determined expressions for $\pi_{m,\mu}$. It may be remarked that the results of Seares are affected by the same scale-error of the *pseudo-visual* scale as the results of Kapteyn and Van Rhijn.

It may be regarded as established that a Gaussian curve such as found by Kapteyn cannot represent the distribution of stellar luminosities. From an inspection of the curve on fig. 3 it appears that a superposition of two distributions may account for the observed form of the curve: the first distribution being roughly a Gaussian with a maximum at $M = +1.5$, the second showing no maximum within the range of M covered by the observations. On the other hand, the distribution of the velocities of the stars and, especially, the change of the apex with the velocity (Strömberg's investigations) seem to point at a duality in the constitution of our stellar universe, the stars belonging partly to our *local universe*, partly — to some *general universe* (background stars); the local universe seems to contain a great proportion of absolutely bright stars, whereas absolutely faint stars possess on the average high velocities and seem to belong in their majority to the general universe. Thus the peculiar shape of the curve on fig. 3 may be explained by a superposition of the luminosity curve of the local universe, with a maximum at +1.5, upon the luminosity curve of the background stars, the majority of which are of low luminosity. Perhaps such a working hypothesis may prove useful in future investigations; from this standpoint it appears necessary to investigate the distribution of luminosities of stars in connection with their space-velocities.